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Ac-driven Nonlinear Schrödinger Equation and Double Sine-Gordon Equation: Numerical Study of Complexes of Localized States

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Abstract. We investigate complexes of localized states in two dynamical systems: (i) directly driven nonlinear Schrödinger equation (NLS) and (ii) double sine-Gordon equation (2SG). Our numerical approach is based on the numerical continuation with respect to the control parameters of the stationary solutions and stability analysis by means of the linearized eigenvalue problem. We show that in the weak damping case the directly driven NLS equation holds stable and unstable multi-soliton complexes. We also show that the second harmonic changes properties and increases the complexity of coexisting static fluxons of 2SG equation.

Keywords: Nonlinear Schrödinger equation, double sine-Gordon equation, soliton, fluxon, stability, bifurcation, numerical continuation

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INTRODUCTION

We study complexes of localized states in two dynamical systems:

• externally-driven nonlinear Schrödinger equation (NLS);
• double sine-Gordon equation (2SG).

Both systems have undergone an extensive mathematical analysis because of their wide range of physical applications. In both cases, our numerical approach is based on numerical continuation of stationary solutions of respective partial differential equations and numerical solution of linearized eigenvalue problems [1, 2] for stability analysis. Numerical continuation algorithm is described in [1, 3, 4]. At each step of numerical continuation, the Newtonian iteration with the 4th order accuracy Numerov’s discretization is utilized. Our aim in this contribution is a numerical study of (i) multi-soliton complexes of ac-driven NLS in the case of weak damping; (ii) multi-fluxon solutions of 2SG depending on the second harmonic.

COMPLEXES IN THE AC-DRIVEN, WEAKLY DAMPED NLS

We consider the nonlinear Schrödinger equation (NLS) driven by a constant external force

\[ i\psi_t + \psi_{xx} + 2|\psi|^2\psi - \psi = -h - i\gamma\psi, \quad \psi_{x}(\pm\infty) = 0, \]

where \( \gamma > 0 \) and \( h > 0 \) are, respectively, dimensionless parameters of the damping strength and the external driving. A number of physical applications of Eq.(1) is listed in [7].

Eq. (1) is the autonomous version of the time-dependent damped externally-driven NLS equation

\[ i\psi_t + \psi_{xx} + 2|\psi|^2\psi = -he^{it} - i\gamma\psi. \]

The driving term of this form can be interpreted as an alternating current (“ac”) driving. Because of this analogy we call the constant driving in Eq.(1) as an “ac-driving” term.

Two types of stationary soliton solutions of Eq.(1) (denoted \( \psi_- \) and \( \psi_+ \) in dependence on the real part sign) are well investigated [5]. Solution \( \psi_+ \) is known to be always unstable while \( \psi_- \)-soliton is stable in some region of the \((h, \gamma)\)-plane. In the case of strong damping \((\gamma > 0.5)\), stationary \( \psi_- \) and \( \psi_+ \) solitons can form multi-soliton complexes.
via the tail-overlap mechanism [6, 1]. Such complexes may be stable like the stable bound state consisting of two \( \psi \)-solitons. In case \( \gamma < 0.5 \), solitons of Eq.(1) decay monotonically and do not have undulations on their tails. Nevertheless, existence of stationary complexes in the absence of damping (\( \gamma = 0 \)) was proved in [7]. Travelling undamped waves and complexes were also obtained in [7, 8]. Two types of travelling solitons are identified: one type is stable for sufficiently low velocities while the second type of solitons stabilizes when travelling faster than a certain critical speed. The stable solitons can form stably travelling bound states.

Our aim is the investigation of multi-soliton complexes in the case of weak damping, \( 0 < \gamma < 0.5 \). Stationary localized solutions of Eq.(1) satisfy the boundary value problem for an ODE:

\[
\psi_{xx} + 2|\psi|^2\psi - \psi = -h - i\gamma \psi, \quad \psi(\pm \infty) = 0. \tag{2}
\]

Let \( \psi_s(x) = \Re + i\Im \) be a stationary solution to the above boundary value problem. Considering a small perturbation in the form \( \psi = \psi_s + [\mu(x) + iv(x)]\exp(\lambda t) \) of solution \( \psi_s \) of Eq.(2) one can obtain the corresponding linearized eigenvalue problem [7]:

\[
\mathcal{H} \bar{\psi} = \lambda \bar{\psi}, \tag{3}
\]

Here \( \bar{\psi} \) is a two-component vector-function \( \bar{\psi}(x) = \begin{pmatrix} \mu \\ v \end{pmatrix} \) and \( J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \). If there is at least one eigenvalue \( \lambda \) with \( \Re \lambda > 0 \), the solution is unstable.

Solutions of Eq.(2) are pathfollowed in \( \gamma \) for the fixed \( h \). As the start point of our numerical continuation, we utilize stationary multi-soliton solutions already known for \( \gamma = 0 \) and for \( \gamma > 0.5 \). Continuation algorithm is characterized by adaptive calculation of the \( h \)-increment and the easy crossing through the turning points [1]. At each step of numerical continuation we apply Newtonian iteration with Newtonian parameter from [9] and the 4th order accuracy Numerov’s discretization. Most calculations have been performed on the interval \( x \in [-100, 100] \) with uniform spatial step-size \( \Delta x = 0.005 \). Stability and bifurcations of stationary solutions of Eq.(1) depending on parameters \( h \) and \( \gamma \) are classified by means of numerical solution of the eigenvalue problem (3). Numerical solution of Eq.(3) is based on Fourier discretization and the resulting algebraic eigenvalue problem is solved numerically using standard EISPACK codes.

At each step of the numerical continuation, we calculate the energy integral and use it as a bifurcation measure in all of calculations:

\[
E = \int \left[ |\psi|^2 + |\psi|^4 - h(\psi + \psi^*) - |\psi_0|^2 + |\psi_0|^4 + h(\psi_0 + \psi_0^*) \right] dx, \quad \psi_0 = \psi(\pm \infty). \tag{4}
\]

**FIGURE 1.** Complex T5 for (a) \( \gamma = 0, h = 0.05 \) and (b) \( \gamma = 0.04, h = 0.05 \)

In [7, 8], we obtained four undamped multi-soliton complexes denoted T2, T3, T4, T5. Only for the complex T5 (see Figure 1(a)), both its real and imaginary part are even, \( i.e. \), only the solution T5 is continuable in nonzero \( \gamma \) [10].
So, the first step of our numerical study is to pathfollow the solution $T5$ to $\gamma > 0$. As this solution is continued, it gradually transforms into the three-soliton complex $\psi_{(---)}$ (see Figure 1(b)). The corresponding $E(\gamma)$ curve turns up at $\gamma = 0.49181$, see Figure 2(a). The separating distance between solitons in the complex is growing to infinity as we continue along the top branch to $\gamma = 0.49$. Both branches on Figure 2(a) have been found to be unstable. The same dependence $E(\gamma)$ is observed for $h = 0.2$. The curve $E(\gamma)$ turns up at $\gamma = 0.16124$, to the branch of the three-soliton complex $\psi_{(-+--)}$ and both complexes are unstable for all values $\gamma$. A representative solution is shown in Figure 2(b) for $\gamma = 0.1, h = 0.2$. So, we have not obtained stable complexes pathfollowing solution $T5$ from $\gamma = 0$.

Next, we continue the strongly damped two-soliton complexes obtained in [1] for $\gamma = 0.52$ to $\gamma < 0.5$. As the starting point of the numerical continuation we take three stationary complexes of two $\psi_-$ solitons with different distances (orbits) between constituents: $\psi_{1,(---)}$, $\psi_{2,(---)}$, $\psi_{3,(---)}$. Here the driving parameter has been fixed $h = 0.35$.

It was already mentioned that in the case $\gamma < 0.5$, solitons of Eq.(1) decay monotonically, i.e., they cannot interact via overlapping of the oscillating tails. Nevertheless, we obtained the same two-soliton complexes that existed for the strong damping case: $\psi_{1,(---)}$, $\psi_{2,(---)}$, and $\psi_{3,(---)}$. They are shown in Figures 3a, 3b, 4a for $\gamma = 0.49, h = 0.35$, respectively. As in the case $\gamma = 0.52$ only the complex $\psi_{2,(---)}$ has been found to be stable. The results of the linear stability analysis were confirmed by means of direct numerical simulations in [11].
As we continue the complex $\psi_{2,(-)}$ from $\gamma = 0.52$ in the direction of decreasing $\gamma$, the curve $E(\gamma)$ turns up to the branch of unstable 4-soliton complex $\psi_{(+--)}$, see Figure 4(b). Continuing $\psi_{2,(-)}$ in the direction of $\gamma > 0.52$ we obtain, after the right-hand side turning point, the unstable complex of two $\psi_1$-solitons which gradually transforms into the flat solution [5]. Resulting $E(\gamma)$ diagram is shown in Figure 5(a). Figure 5(b) is a detailed enlargement of the area around the left-side turning point. So, in case $h = 0.35$, the complex $\psi_{2,(-)}$ exists and is stable between the two turning points: the left-hand side turning point $\gamma = 0.489806$ and the right-hand side turning point $\gamma = 0.55$.

Taking smaller value of $h$ one can obtain two-soliton complex for lower values of $\gamma$. One can expect that there is a corridor of values $h$ and $\gamma$ on the $(h, \gamma)$-plane where the weakly damped complex $\psi_{2,(-)}$ exists and is stable. Say, in case $h = 0.32$ the left-hand side turning occurs at $\gamma \approx 0.449$. The stable complex $\psi_{2,(-)}$ occurring for $h = 0.32$, $\gamma = 0.45$ (result of direct numerical simulation) is shown in Figure 6. For direct numerical simulation of Eq.(1) we employed the split-step method [12] with Fourier discretization in space.
DOUBLE SINE-GORDON EQUATION: EFFECT OF THE 2ND HARMONIC

We consider the double sine-Gordon equation (2SG) in the following form:

\[ \phi_{xx} - \phi_{tt} = a_1 \sin \phi + a_2 \sin 2\phi - \gamma, \quad t > 0, \quad x \in (-l, l), \quad \phi_{x}(\pm l, t) = h_{e}. \quad (5) \]

One of the most popular applications of Eq.(5) is the long Josephson junction (LJJ) model where 2SG equation describes the magnetic flux distributions in the case of a finite length overlap contact, see [2] and references therein. In this framework, \( \phi \) is the magnetic flux distribution, \( h_{e} \) – external magnetic field, \( \gamma \) – external current, \( l \) – semilength of the junction, \( a_1 \) and \( a_2 \) – parameters of contribution of the first and second harmonics in the current-phase relation. All parameters are considered to be dimensionless. The sign of \( a_2 \) can be positive or negative, depending on the mechanism of suppression of superconducting state in electrodes. In this paper we consider only the case \( a_2 \leq 0 \).

The 2nd harmonic plays an important role in the formation of the magnetic flux distribution in LJJ, see, e.g., [13, 14] and references therein. Our aim is to investigate complexity of multi-fluxon static distributions of (5) depending on the parameters \( a_2 \) and \( h_{e} \). All other parameters have been fixed \( a_1 = 1, \gamma = 0, 2l = 10 \).

The static magnetic flux distributions satisfy the following boundary value problem:

\[ -\phi_{xx} + a_1 \sin \phi + a_2 \sin 2\phi - \gamma = 0, \quad x \in (-l, l), \quad \phi_{x}(\pm l) = h_{e}. \quad (6) \]

To perform the stability analysis of stationary solutions \( \phi_{s} \) of (6) we linearize (5) about the stationary solution with small perturbation of the form \( \phi = \phi_{s} + \psi(x) \exp(-\sqrt{\lambda}t) \). The resulting eigenvalue problem is [15]:

\[ -\psi_{xx} + q(x)\psi = \lambda \psi, \quad q(x) = a_1 \cos \phi + 2a_2 \cos 2\phi. \quad (7) \]

Supplying Eq.(7) with appropriate boundary conditions and normalization condition

\[ \psi_{x}(\pm l) = 0, \quad \int_{-l}^{l} |\psi(x)|^2 \, dx = 1, \quad (8) \]

we obtain the Sturm-Liouville problem (7,8). The case of \( \lambda_0 > 0 \) corresponds to the stable solution \( \phi \) where \( \lambda_0 \) is the minimal eigenvalue of Eq.(7). In the case of \( \lambda_0 < 0 \), solution \( \phi \) is unstable. The case \( \lambda_0 = 0 \) indicates the bifurcation with respect to one of parameters of Eq.(5).
Eqs.(6-8) are considered as the unified system with respect to unknowns \( \varphi(x) \), \( \psi(x) \) and \( \lambda = \lambda_0 \). Numerical solution was performed by means of Newtonian iteration, Numerov’s 4th order finite difference approximation and the continuation technique described in [1, 3]. (Note, by putting \( \lambda = 0 \) one can modify the Newtonian iterative process in order to calculate critical values of \( \gamma, h_c \) or another parameter together with the unknown functions \( \varphi(x) \), \( \psi(x) \) at the bifurcation points.)

During the numerical continuation we calculate two quantities to characterize solutions:

- the full magnetic flux
  \[
  \Delta \varphi = \varphi(l) - \varphi(-l),
  \]
- the “number of fluxons”
  \[
  N = \frac{1}{2\pi} \int_{-l}^{l} \varphi(x) \, dx.
  \]

Taking into account the second harmonic, i.e., introducing a nonzero \( a_2 \), changes the properties of the standard static magnetic flux distributions and gives rise to new (stable and unstable) static solutions [2, 13, 16, 17, 18]. Two basic distributions are known at \( a_2 = h_c = 0 \): the constant Meissner solution \( M_0 \) with \( N[M_0] = 0 \) and the fluxon solution \( \Phi^1 \) with \( N[\Phi^1] = 1 \). In continuation of \( M_0 \) to nonzero \( a_2 \) and/or \( h_c \), the Meissner solution loses its uniformity and gets fluxon-like form. In [13] this state was called “small fluxon” while standard fluxon \( \Phi^1 \) denoted as “large fluxon”.

Below, when present results for nonzero \( a_2 \), we also utilize the same notation \( \Phi^1 \equiv \Phi^1_{\text{large}}, M_0 \equiv \Phi^1_{\text{small}} \).

When \( a_2 < -0.5 \) one more solution is appearing, namely, the constant solution \( M_{\text{occ}} [2] \).

Let us compare the \( h_c \)-dependence of the magnetic flux \( \Delta \varphi/2\pi \) in two cases: \( -0.5 \leq a_2 \leq 0 \) and \( a_2 < -0.5 \). As representative values of \( a_2 \) we take \( a_2 = 0 \) and \( a_2 = -0.7 \).

In both cases we use solutions known for \( h_c = 0 \) as the starting points of our continuation process to nonzero \( h_c \).

In Figure 7, we plot the normalized rate of change of the magnetic flux \( \Delta \varphi/2\pi \) versus the external magnetic field \( h_c \) for the case \( a_2 = 0 \) to demonstrate the connection between the coexisting stable and unstable solutions. We obtained two curves, the first of which connects \( M_0 \) (\( h_c = 0 \)) to multi-fluxon states \( \Phi^N \) with even “number of fluxons” \( N \) and the other one connects \( \Phi^1 \) (\( h_c = 0 \)) to solutions \( \Phi^N \) with odd \( N \). Both curves are snaking up as the “number of fluxons” increases. Here and below, stable and unstable solutions are plotted using solid and dashed lines, respectively.

Light circles indicate turning points; solid circles show the points where stability changes. It is shown that number of coexisting multi-fluxon states is increasing as \( h_c \) is growing. Indeed, there are only two solutions at \( h_c = 0 \) while at \( h_c = 1 \) we have three stable solutions (solution \( \Phi^0 \), one- and two-fluxon states \( \Phi^1, \Phi^2 \)) coexisting with unstable two-fluxon complex \( \Phi^2 \).

A pair of additional three-fluxon solutions (stable \( \Phi^3 \) and unstable \( \Phi^3 \)) appears at \( h_c = 2 \), etc.

While \( a_2 \) is between 0 and \( -0.5 \) the dependence \( \Delta \varphi/2\pi(h_c) \) stays qualitatively the same [17]. However, when we take \( a_2 < -0.5 \) the complexity of coexisting branches increases. The representative case of \( a_2 = -0.7 \) is shown in

\[\text{FIGURE 7.} \quad \text{Diagram } \Delta \varphi/2\pi(h_c) \text{ for } a_2 = 0, a_1 = 1, 2l = 10, \gamma = 0 \text{ (reproduced from [3])}\]
FIGURE 8. Coexisting stationary solutions of Eq.(6) for $a_2 = -0.7, a_1 = 1, 2l = 10, \gamma = 0$ at $h_e = 0$

Figures 8-11. Even at the start point $h_e = 0$, instead of two solutions for $a_2 = 0$ we have three solutions: unstable $\Phi^1_l = \Phi^1_{large}, M_0 = \Phi^1_{small}$, and stable $M_{ac}$, see Figure 8. For visual clarity, we plot the 1st derivative of solutions.

FIGURE 9. Diagram $\Delta \phi / 2\pi(h_e)$ for $a_2 = -0.7, a_1 = 1, 2l = 10, \gamma = 0$

Similarly to the case of $a_2 = 0$ [3] and $a_2 = -0.5$ [17], we plot two curves, the first of which connects $\Phi^1_{small}$ ($h_e = 0$) to the multi-fluxon states $\phi^N$ with even “number on fluxons” $N$ and the other one connects $\Phi^1_{large}$ ($h_e = 0$) to the solutions $\phi^N$ with odd $N$ (Figure 9). In addition we have found a short branch connecting $M_{ac}$ and $\Phi^1_{small}$ which does not exist in the case $|a_2| \leq 0.5$. This branch is seen in details in Figure 10.

When $h_e$ is growing, “small” and “large” fluxons stabilize and complexity of coexisting solutions increases. Indeed, Figure 11 demonstrates four stable multi-fluxon solutions coexisting at $h_e = 1.5$ with two unstable solutions $\tilde{\phi}^2$ and $\tilde{\phi}^3$ (not plotted).

Note one more difference between diagrams on Figure 7 and Figure 9: it is the appearing of unstable intervals $\tilde{\phi}^N$ between stable portions $\phi^N$ and $\phi^{N*}$ in Figure 9. One can expect that the further increase of $|a_2|$ can lead to overlapping of stability domains of $\phi^N$ and $\phi^{N*}$, i.e., coexisting of different states with the same “number of fluxons”. Indeed, it was shown in [18] that stability domains of $\phi^1$ and $\phi^{1*}$ overlap when $a_2 < -0.8$.

We also have to mention that our diagram in Figure 10 does not include mixed bound states of “small” and “large” fluxons. This type of solution (see Figure 12) was found in [17] for $a_2 = -1$ but its dependence on $h_e$ and $a_2$ has not
FIGURE 10. Diagram $\Delta \phi/2\pi(h_e)$ for $a_2 = -0.7, a_1 = 1, 2l = 10, \gamma = 0$ around $h_e = 0$ yet been analyzed.

FIGURE 11. Coexisting stable stationary solutions of Eq.(6) for $a_2 = -0.7, a_1 = 1, 2l = 10, \gamma = 0$ at $h_e = 1.5$

SUMMARY

Complexes of localized states are numerically analyzed in the directly driven nonlinear Schrödinger equation and the double sine-Gordon equation. In both cases the numerical approach is based on the numerical continuation with respect to the control parameters of the stationary solutions and stability and bifurcation analysis of the respective linearized eigenvalue problem. Our numerical approach can be applied to a broad class of nonlinear equations and can be easily generalized for systems of PDEs as well.

Multi-soliton complexes of the NLS equation are studied in the undamped and the weak damping regimes. We show that in the weak damping case, in spite of the absence of undulations in tails, the directly driven NLS equation supports stable and unstable multi-soliton complexes. The results are confirmed by means of direct numerical simulations of the time-dependent NLS equation. Properties of the multi-fluxon solutions of the 2SG equation are studied depending on the parameter of the second harmonic. We show that the second harmonic changes properties and increases the
complexity of coexisting static fluxons of the 2SG equation. Results are discussed within the frame of the long Josephson junction model.

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