Numerical investigation of the second harmonic effects in the LJJ

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Abstract. We study the long Josephson junction (LJJ) model which takes into account the second harmonic of the Fourier expansion of Josephson current. The sign of second harmonic is important for many physical applications. The influence of the sign and value of the second harmonic on the magnetic flux distributions is investigated. At each step of numerical continuation in parameters of the model, the corresponding nonlinear boundary problem is solved on the basis of the continuous analog of Newton's method with the 4th order Numerov discretization scheme. New solutions which do not exist in the traditional model have been found. The influence of the second harmonic on stability of magnetic flux distributions for main solutions is investigated.

Key words: long Josephson junction, Sturm-Liouville, double sine-Gordon, bifurcation, continuous analog of Newton's method, fluxon, Numerov's finite-difference approximation

1 Formulation of the problem

Physical properties of magnetic flux in Josephson junctions (JJs) play important role in the modern superconducting electronics. Tunnel SIS JJs are known to be having the sinusoidal current phase relation. However, the decrease of the barrier transparency in the SIS JJs leads the deviations of the current-phase relation from the sinusoidal form [1].

We study the static magnetic flux distributions in the long JJs taking into account the second harmonic in the Fourier-decomposition of the Josephson current. This model is described by the double sine-Gordon equation (2SG) for magnetic flux distribution in the static regime [2],[3],[4]:

$$-\varphi'' + a_1 \sin\varphi + a_2 \sin 2\varphi - \gamma = 0, \quad x \in (-l; l), \tag{1}$$

with the boundary conditions in the the following form

$$\varphi'(\pm l) = h_e. \tag{2}$$

Here and below the prime means a derivative with respect to the coordinate x. The magnitude γ is the external current, l is the semilength of the junction, a_1

and a_2 are parameters corresponding the contribution of 1st and 2nd harmonic, respectively. h_e is external magnetic field. All the magnitudes are dimensionless.

The sign of the second harmonic depends on a physical application under study. It is important, in particular, in junctions like SNINS and SFIFS, where N is a normal metal and F is a weak metallic ferromagnet [5]. Interesting properties of long Josephson junctions with an arbitrarily strong amplitude of the second harmonic in current phase relation were considered in [6]. We investigate the existence and stability magnetic flux distributions in dependence on the second harmonic contribution in both cases of negative and positive sign.

Stability analysis of $\varphi(x,p)$ is based on numerical solution of the corresponding Sturm-Liouville problem [7,8]:

$$-\psi'' + q(x)\psi = \lambda\psi, \quad \psi'(\pm l) = 0 \tag{3}$$

with a potential $q(x) = a_1 \cos\varphi + 2a_2 \cos 2\varphi$.

The minimal eigenvalue $\lambda_0(p) > 0$ corresponds the stable solution. In case $\lambda_0(p) < 0$ solution $\varphi(x,p)$ is unstable. The case $\lambda_0(p) = 0$ indicates the bifurcation with respect to one of parameters $p = (l, a_1, a_2, h_e, \gamma)$.

2 Numerical scheme

For numerical solution of the boundary problem (1),(2) we apply an iteration algorithm based on the continuous analog of Newton's method (CANM) [8]. Let an initial approximation for $\varphi_0(x)$ be given. At k^{th} step $(k = 1, 2, , \ldots)$ we calculate:

1. Iteration correction $w_k(x)$ by solving linearized boundary problem

$$-w''_k + q_{k-1}(x)w_k = \varphi''_{k-1} - f_{k-1}(x)\,, \tag{4}$$

$$w'_k(-l) = -\varphi'_{k-1}(-l) + h_e\,, \tag{5}$$

$$w'_k(l) = -\varphi'_{k-1}(l) + h_e\,, \tag{6}$$

where $f(x) = a_1 \sin\varphi + a_2 \sin 2\varphi - \gamma$.

2. Next approximation

$$\varphi_k(x) = \varphi_{k-1}(x) + \tau_k\, w_k$$

where parameter τ_k is calculated by the Ermakov-Kalitkin formula [9].

Further in order to simplify notations the iteration indices are omitted.

We introduce the grid $M_h = \{x_i = -l + (i-1)h, i = \overline{1,N}, x_N = l, h = 2l/(N-1)\}$. Numerov's discrete approximation [10] of (4)–(6) yields the following linear algebraic system with the three-diagonal structure at $i = \overline{3, N-2}$:

$$-25w_1 + 48w_2 - 36w_3 + 16w_4 - 3w_5 = 12h(h_e - \varphi'_1)$$

$$a_2w_1 + b_2w_2 + c_2w_3 + d_2w_4 + e_2w_5 = r_2$$

x	$h=0.15625$	$h=0.078125$	$h=0.0390625$	$\sigma\approx$
-5.00	0.0539477770043	0.0539492562101	0.0539493470654	16.2809
-3.75	0.1018425717580	0.1018436848799	0.1018437558002	15.6954
-2.50	0.3299569440243	0.3299543520791	0.3299541941853	16.4157
-1.25	1.1169110352657	1.1169428501398	1.1169448259542	16.1022
1.25	5.1662742719134	5.1662424570391	5.1662404812249	16.1022
2.50	5.9532283631551	5.9532309551002	5.9532311129941	16.4157
3.75	6.1813427354214	6.1813416222995	6.1813415513793	15.6954
5.00	6.2292375301752	6.2292360509694	6.2292359601142	16.2809

Table 1. Values of function φ and quantities σ (7) in some points of the interval $[-l;l]$ for solution of kind Φ^1 at $2l=10$, $\gamma=0$, $h_e=0$, $a_1=1$, $a_2=0$.

$$a_i w_{i-1} + b_i w_i + c_i w_{i+1} = r_i\,, \quad i=\overline{3,N-2}\,,$$
$$a_{N-1}w_N + b_{N-1}w_{N-1} + c_{N-1}w_{N-2} + d_{N-1}w_{N-3} + e_{N-1}w_{N-4} = r_{N-1}$$
$$25w_N - 48w_{N-1} + 36w_{N-2} - 16w_{N-3} + 3w_{N-4} = 12h(h_e - \varphi'_N)$$

where the coefficients are determined by the following way

$$a_2 = 1\,,\ b_2 = -2 - \frac{7h^2}{6}\,q_2\,,\ c_2 = 1 + \frac{5h^2}{12}\,q_3\,,\ d_2 = -\frac{h^2}{3}\,q_4\,,\ e_2 = \frac{h^2}{12}\,q_5\,,$$
$$r_2 = \frac{h^2}{12}\,(14f_2 - 5f_3 + 4f_4 - f_5) - \frac{h^2}{12}\,(14\varphi''_2 - 5\varphi''_3 + 4\varphi''_4 - \varphi''_5)\,,$$
$$a_i = 1 - \frac{h^2}{12}\,q_{i-1}\,,\ b_i = -2 - \frac{5h^2}{6}\,q_i\,,\ c_i = 1 - \frac{h^2}{12}\,q_{i+1}\,,\ i=\overline{3,N-2}\,,$$
$$r_i = \frac{h^2}{12}\,(f_{i+1} + 10f_i + f_{i_1}) - \frac{h^2}{12}\,(\varphi''_{i+1} + 10\varphi''_i + \varphi''_{i_1})\,,\ i=\overline{3,N-2}\,,$$
$$a_{N-1} = 1\,,\ b_{N-1} = -2 - \frac{7h^2}{6}\,q_{N-1}\,,\ c_{N-1} = 1 + \frac{5h^2}{12}\,q_{N-2}\,,$$
$$d_{N-1} = -\frac{h^2}{3}\,q_{N-3}\,,\ e_{N-1} = \frac{h^2}{12}\,q_{N-4}\,,$$
$$r_{N-1} = \frac{h^2}{12}\,(-f_{N-4} + 4f_{N-3} - 5f_{N-2} + 14f_{N-1}) -$$
$$-\frac{h^2}{12}\,(-\varphi'' + 4\varphi''_{N-3} - 5\varphi''_{N-2} + 14\varphi''_{N-1})\,,$$

where $\varphi_i = \varphi(x_i)$, $q_i = q(x_i)$, $f_i = f(x_i)$.

In order to test the accuracy order of the above numerical scheme we perform the calculations of (1),(2) at the sequence of uniform grids with steps h, $h/2$ and $h/4$ ($h = 0.15625$). The results for solutions of the kind Φ^1 are presented in the table 1. It is seen, the quantities σ calculated by formula

$$\sigma(x_i) = \frac{\varphi_h(x_i) - \varphi_{h/2}(x_i)}{\varphi_{h/2}(x_i) - \varphi_{h/4}(x_i)}\,, \quad i = 1,2,\ldots,N, \tag{7}$$

are close to 2^4 that corresponds the theoretical accuracy order $O(h^4)$ of Numerov's approximation.

The StLP (3) is approximated by the three-point finite-difference second order formulas [11]. The resulting algebraic eigenvalue problem is solved numerically with help of a standard subroutine [12].

3 Numerical results and conclusions

Trivial solutions. In the "traditional" case $a_2 = 0$ two trivial solutions $\varphi = 0$ and $\varphi = \pi$ (denoted by M_0 and M_π respectively) are known at $\gamma = 0$ and $h_e = 0$. Accounting of the second harmonic $a_2 \sin 2\varphi$ leads to appearing two additional solutions $\varphi = \pm \arccos(-a_1/2a_2)$ (denoted as $M_{\pm ac}$). The corresponding λ_0 as functions of 2SG-equation coefficients have the form $\lambda_0[M_0] = a_1 + 2a_2$, $\lambda_0[M_\pi] = -a_1 + 2a_2$ and $\lambda_0[M_{\pm ac}] = (a_1^2 - 4a_2^2)/2a_2$. The exponential stability of these constant solutions (CS) is determined by the signs of the parameters a_1 and a_2 and by its ratio a_1/a_2.

The full energy associated with the distribution of $\varphi(x)$ is calculated by the formula [8]

$$F(p) = \int_{-l}^{l} \left[\frac{1}{2} \varphi'^2 + 1 - q(x) - \gamma\varphi \right] dx - h_e \Delta\varphi.$$

The full energy behavior in dependence on a_2 for considered distributions in the junction at $h_e = 0$, $\gamma = 0$, $a_1 = 1$, $2l = 10$ is shown in Fig. 1.

Stability properties of trivial solutions have been investigated in [13].

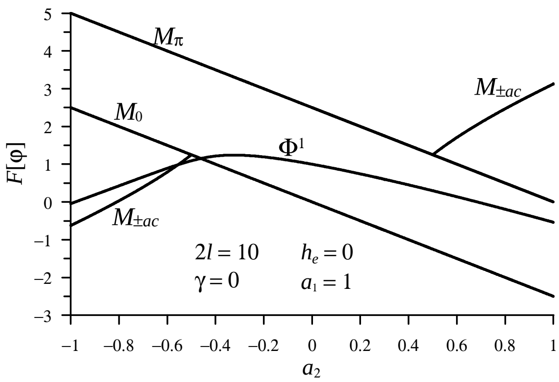

Fig. 1. Full energy in dependence on $a_2 \in [-1; 1]$ at $h_e = 0$, $\gamma = 0$ and $2l = 10$ for CS and Φ^1.

Fluxon solutions. The fluxons play an important role in the JJ physics. At small external fields h_e such distributions are fluxon Φ^1, antifluxon Φ^{-1} and

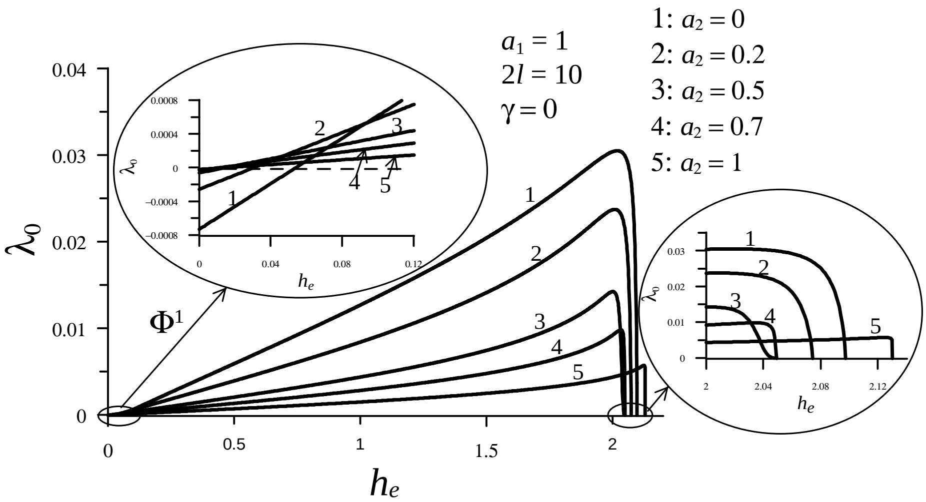

Fig. 2. Dependence $\lambda_0(h_e)$ for Φ^1 at increase of $a_2 \in [0;1]$ and $a_1 = 1$, $2l = 10$, $\gamma = 0$.

their bound states $\Phi^1\Phi^{-1}$ and $\Phi^{-1}\Phi^1$. As external magnetic field h_e is growing, more complicated stable fluxon and bound states appear: $\Phi^{\pm n}$ and $\Phi^{\pm n}\Phi^{\mp n}$ $(n = 1, 2, 3, ...)$.

The energy of one-fluxon distribution Φ^1 limits to unit $F(a_2 \to 0) \to 1$ which corresponds to an energy of a single fluxon Φ^1_∞ in a traditional "infinite" junction model at $a_1 = 1$, $a_2 = 0$. With change of a_2 the number of fluxons [8]

$$N(p) = \frac{1}{2l\pi} \int_{-l}^{l} \varphi(x)\, dx,$$

corresponding to the distribution Φ^1 is conserved i.e. $\partial N/\partial a_2 = 0$. Here we have $N[\Phi^1] = 1$.

Let us discuss the main features of the dependence $\lambda_0(h_e)$ for one-fluxon state Φ^1 in two intervals: $a_2 \in [0, 1]$ and $a_2 \in [-1, 0]$.

The change of the curve $\lambda_0(h_e)$ for one-fluxon state Φ^1 when the parameter a_2 increases in the interval $a_2 \in [0; 1]$ is shown in Fig. 2. When $h_e = 0$, the state Φ^1 remains unstable. With increase in a_2, λ_0 increases monotonically and tends to zero. With increase in magnetic field this solution becomes stable. The bifurcation point moves to the left with increase of parameter in the interval $a_2 \in [0; 0.7]$. At $a_2 > 0.7$ it goes to the right again. It follows a stability interval ending at $h_{cr} \approx 2$. The second bifurcation point also moves to the left at $a_2 \in [0; 0.5]$. When a_2 is increased in $a_2 \in [0.5; 1]$, the bifurcation value h_{cr} is also increased.

In the interval $a_2 \in [-1, 0]$ we observe the following. When a_2 increases in $(-0.5; 0]$, the curve $\lambda_0(h_e)$ for Φ^1 moves to the right (Fig. 3). At $a_2 < -0.5$ (case

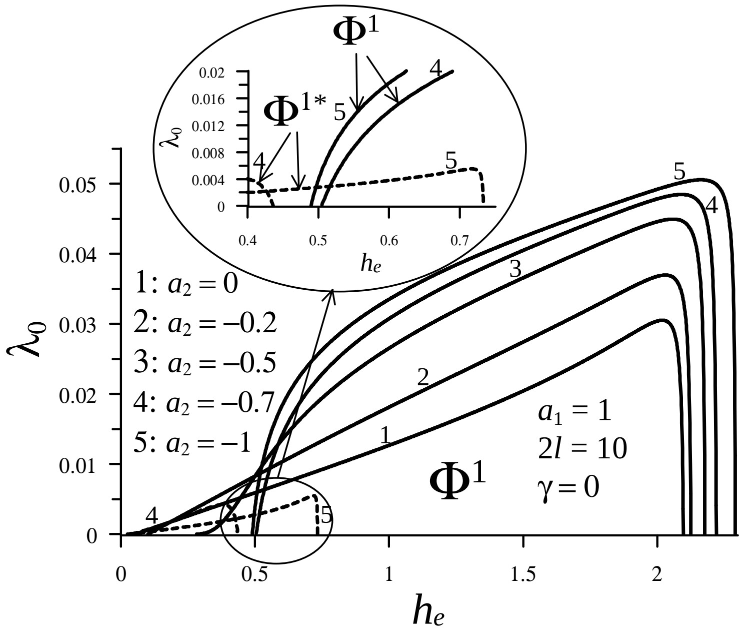

Fig. 3. $\lambda_0(h_e)$ in dependence on $a_2 \in [-1; 0]$ for Φ^1 and Φ^{1*}, at $a_1 = 1$, $2l = 10$, $\gamma = 0$.

$a_2 = -0.7$ in Fig. 3), the curve corresponding to the stable solution Φ^1 has two separate branches that are intersected at $a_2 \approx -0.8$). Here we observe a region along h_e, where two different stable one-fluxon solutions (denoted by Φ^1 and Φ^{1*}) coexist, see Fig. 4.

Thus, we considered both positive and negative contributions of the second harmonic in 2GS equation. It is shown that its accounting leads appearing new constant solutions and changes the stability properties of the fluxon solutions. Coexisting of two stable one-fluxon solutions requires further analysis and physical interpretation.

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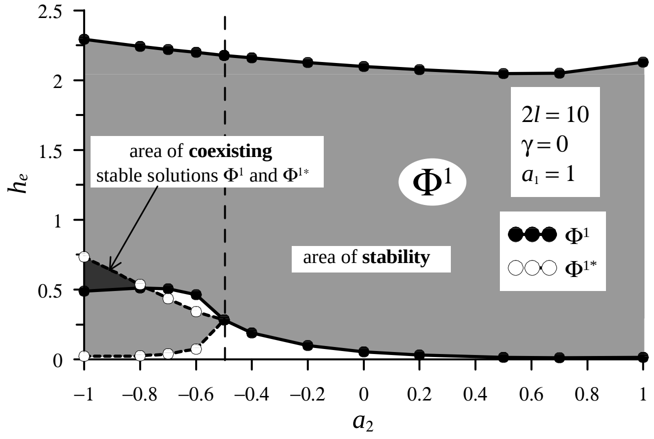

Fig. 4. Bifurcation diagram of one-fluxon solutions at the plain of parameters a_2 and h_e. Here $a_1 = 1$, $2l = 10$, $\gamma = 0$.

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