Interconnection between static regimes in the LJJs described by the double sine-Gordon equation

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Abstract. The second harmonic contribution to the current-phase relation changes the properties of the static magnetic flux distributions in the long Josephson junction (LJJ) and inspires new homogenious and fluxon static states. We study stability properties and bifurcations of these static regimes within the frame of a model described by the double sine-Gordon equation. The critical curves behavior and the interconnection between different types of magnetic flux distributions are analyzed.

1. Motivation and model

Long Josephson junctions (LJJs) are attractive objects of theoretical and experimental investigations because of a wide range of applications in nano- and quantum physics. Important step of their experimental study is the measurement of *static* properties including dependence of the critical current on the external magnetic field [1].

In this paper we investigate the static magnetic flux distributions in LJJ's model described by the boundary value problem for the double sine-Gordon equation (2SG) [2]:

$$-\varphi'' + a_1 \sin \varphi + a_2 \sin 2\varphi - \gamma = 0, \quad x \in (-l; l), \quad \varphi'(\pm l) = h_e. \quad (1)$$

Here the prime means a derivative with respect to the coordinate x, γ is the external current, l – the semilength of the junction, a_1 and a_2 – the normalized amplitudes of the first and second harmonics of the Josephson current [3, 4], h_e – the external magnetic field. All the magnitudes are assumed to be dimensionless [2].

As previously in [5, 6, 7, 8, 9], we analyze stability and bifurcations of static solutions $\varphi(x, p)$, $p = (l, a_1, a_2, h_e, \gamma)$ on the basis of numerical solution of the corresponding Sturm-Liouville problem [10]:

$$-\psi'' + q(x)\psi = \lambda\psi, \quad \psi'(\pm l) = 0, \quad q(x) = a_1 \cos \varphi + 2a_2 \cos 2\varphi, \quad \int_{-l}^{+1} [\psi(x)]^2 \, dx = 1. \quad (2)$$

The minimal eigenvalue $\lambda_0(p) > 0$ corresponds to the stable solution. In case $\lambda_0(p) < 0$ solution $\varphi(x, p)$ is unstable. The case $\lambda_0(p) = 0$ indicates the bifurcation with respect to one of the parameters p. Details of numerical approach are given in [5, 8].

In order to characterize the solutions of Eq.(1) we calculate two quantities:

- full magnetic flux of the distribution $\Delta\varphi = \varphi(l) - \varphi(-l)$;
- quantity $N(p) = \frac{1}{2l\pi}\int_{-l}^{l}\varphi(x)\,dx$ denoted as "number of fluxons" in [5].

The second harmonic contribution plays important role in the modeling of the SNINS and SFIFS junctions (where S is a superconductor, I is an insulator, N is a normal metal and F is a weak metallic ferromagnet) [3, 11]. The second harmonic factor in the LJJ current-phase relation deforms the shape and changes the properties of standard static magnetic flux distributions. Also, the nonzero a_2 inspires new homogenious and fluxon static states [1, 5, 9, 12].

In this contribution we study the effect of second harmonic factor a_2 on the interconnection between static magnetic field distributions. We show that nonzero a_2 gives a rise the stable mixed states of fluxons and leads to the "created-by-current" (CBC) states.

Here we present results for the case of negative factor a_2 only.

2. Results and discussion
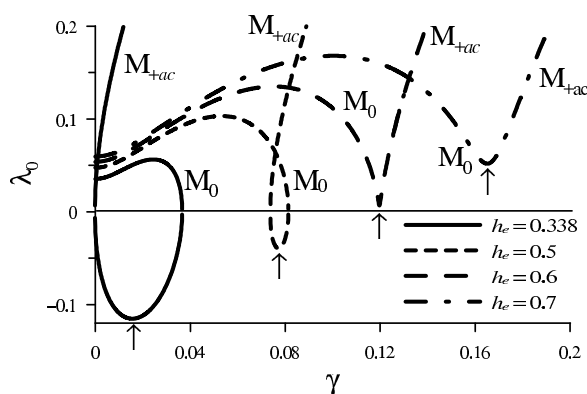
Figure 1. Dependence $\lambda_0(\gamma)$ for M_0 and M_{+ac} for $h_e \in [0.338; 0.7]$ and $a_1 = 1$, $a_2 = -0.7$, $2l = 10$. Vertical rows indicate the points of transformation of the branch M_0 to M_{+ac}.
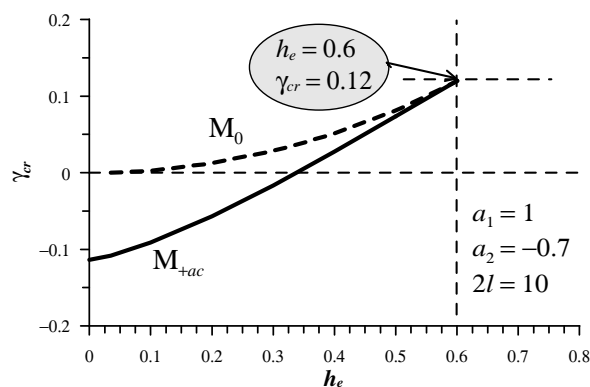
Figure 2. Fragment of $\gamma_{cr}(h_e)$ diagram: CBC-type critical curve for M_{+ac} distribution. Here $a_1 = 1$, $a_2 = -0.7$, $2l = 10$.

Together with the well-known distributions (standardly called M_π and M_0) the nonzero a_2 in Eq.(1) gives a rise another state (denoted as $M_{\pm ac}$ in [5]). Stability and bifurcations of $M_{\pm ac}$ have been investigated in [5, 9]. The states M_0, M_π and $M_{\pm ac}$ are constant for $h_e = 0$ and lose their uniformity as h_e is nonzero.

Interesting property of the a_2-inspired $M_{\pm ac}$ distributions is that for some values h_e they are unstable at $\gamma = 0$ and stabilize under nonzero current through the junction. This type of distribution was observed in LJJs with inhomogeneities [13] and denoted as "created-by-current" state (CBC). Let us demonstrate the CBC-properties of M_{+ac} state for the case $a_2 = -0.7$.

The M_{+ac} distribution is stable for $\gamma = 0$ as h_e is sufficiently small. However, as h_e is growing to $h_e > 0.338$ the solution M_{+ac} loses its stability at $\gamma = 0$ and stabilizes only for $\gamma = \gamma_{cr} > 0$. Corresponding dependence of minimal eigenvalue λ_0 on γ for several values h_e is given in Fig.1. It is seen that the curves $\lambda_0(\gamma)$ for M_0 and for M_{+ac} are interconnected. In the case $h_e = 0.5$, the stable portions of M_0- and M_{+ac}-curves overlap, i.e. stable M_0 and M_{+ac} states coexist in a narrow interval of h_e. For $h_e > 0.6$ the state M_0 smoothly transforms to M_{+ac}-distribution without a loss of stability, i.e. whole curve $M_0 \leftrightarrow M_{+ac}$ is stable. So, the CBC-type M_{+ac} state is observed at $0.338 < h_e < 0.6$.

Corresponding critical curves $\gamma_{cr}(h_e)$ of M_0 and M_{+ac} distributions are shown in Fig.2. Dashed curve shows the bifurcation points $\lambda_0 = 0$ of the branches M_0 on Fig.1 (these points are

placed on the right of vertical rows). Solid curve in Fig.2 corresponds the points $\lambda_0 = 0$ on the left of vertical rows (see Fig.1) which belong the branches M_{+ac}. The solid and dashed critical curves in Fig.2 meet at the point $h_e = 0.6$, $\gamma_{cr} = 0.12$ where the state M_0 transforms to the M_{+ac} just at the bifurcation point $\lambda_0 = 0$, see the long-dashed curve on Fig.1.

Critical curves for M_{+ac} and φ^n ($n = 1, 2, 3, 4, 5$) distributions in the case $a_1 = 1$, $a_2 = -0.7$, $2l = 10$ are shown in Fig.3. The curve demarcated M_{+ac} corresponds the critical points (h_e, γ_{cr}) where the solution M_{+ac} loses its stability as we continue the top branches $\lambda_0(h_e)$ (see Fig.2) in the growing h_e direction. We stress the strong overlapping of critical curves corresponding to the fluxon states φ^1, φ^2, φ^3, *etc.* as a result of the second harmonic contribution and the sufficiently large length of the junction. This means, at some values h_e we have two or more critical values of current where coexisting branches of fluxon distributions have a rise.
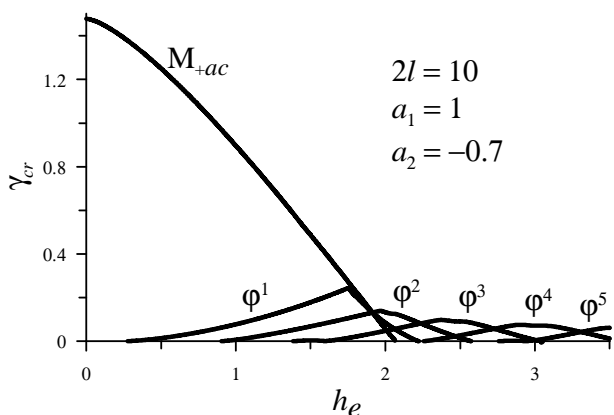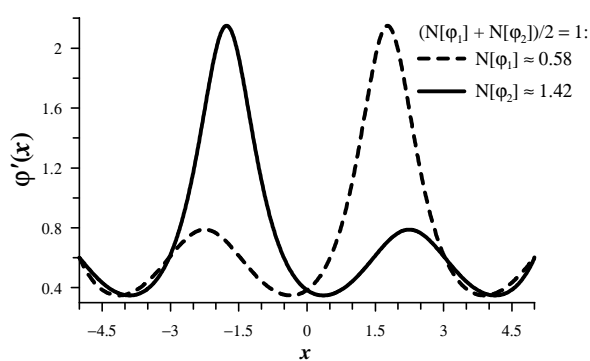

Figure 3. Critical curves for M_{+ac} and φ^n ($n = 1, 2, 3, 4, 5$) distributions. Here $a_1 = 1$, $a_2 = -0.7$, $2l = 10$.
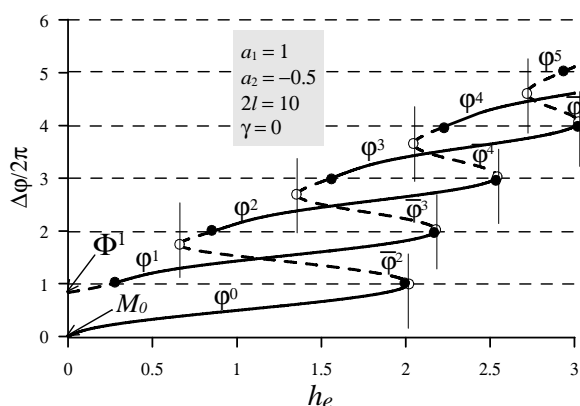
Figure 4. Full magnetic flux in dependence on h_e for M_0 and φ^n, $n \geq 1$; $a_2 = -0.5$, $a_1 = 1$, $2l = 10$, $\gamma = 0$.
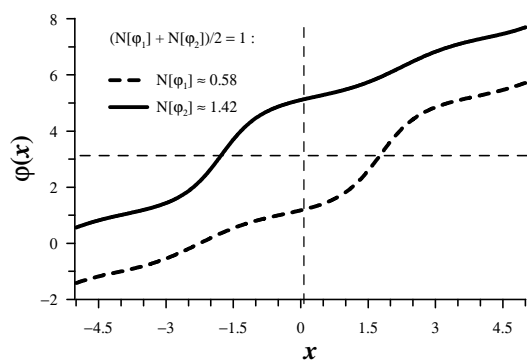
Figure 5. The magnetic flux distribution $\varphi(x)$ for the mixed stable bound state of fluxons with $h_e = 0.6$, $a_1 = 1$, $a_2 = -1$, $2l = 10$, $\gamma = 0$.

Figure 6. The internal magnetic field distribution $\varphi'(x)$ for the mixed stable bound state of fluxons with $h_e = 0.6$, $a_1 = 1$, $a_2 = -1$, $2l = 10$, $\gamma = 0$.

In our recent paper [8], we established, for $a_2 = 0$, the interconnection between the trivial solution $M_0|_{h_e=0}$ and the static fluxon distributions φ^n with even numbers n as well as the interconnection between the fluxon state $\Phi^1 = \varphi^1|_{h_e=0}$ and φ^n with odd numbers n. The same

effect is observed in the case $a_2 \neq 0$. Interconnection between coexisting (stable and unstable) static distributions of Eq.(1) is demonstrated in Fig.4 where the full magnetic flux distribution $\Delta\varphi/2\pi$ is shown in dependence on h_e for $a_2 = -0.5$, $a_1 = 1$, $2l = 10$, $\gamma = 0$.

The $\Delta\varphi(h_e)$ branches in Fig.4 are obtained by numerical continuation of the solutions $M_0|_{h_e=0}$ and $\Phi^1 = \varphi^1|_{h_e=0}$ from the point $h_e = 0$ to the $h_e > 0$ direction. Stable and unstable branches are plotted, respectively, by solid and dashed curves. At the light circle points the curves $\Delta\varphi(h_e)$ turn back to another, upper, branches. The change of stability occurs at the dark circle points where the $\lambda_0(h_e)$ crosses zero. When the $\Delta\varphi(h_e)$ curve turns to the left ("$\supset$"-type turning) the quantity N is increased to $N + 2$. So, the branch started from the $M_0|_{h_e=0}$ solution, joins the fluxons with the even N while the another branch (emanating from Φ^1 fluxon) connects fluxons with the odd quantity N.

The $\Delta\varphi(h_e)$ diagram in Fig.4 looks similar the corresponding diagram for $a_2 = 0$ in [8]. The further increasing of the a_2 contribution produces more complex interconnection between coexisting branches.

Beside of the mentioned above M_{ac} solution, one more a_2-inspired distribution is known, so called "small fluxon" [12, 9]. While the standard "large fluxon" Φ^1 has $N = 1$, the "small fluxon" is characterized by $N = 0$. Another a_2-inspired fluxon solution (denoted as Φ^{1*} in [7]) exists at $a_2 < -0.5$ and characterized by $N = 1$. Beside, our study shows that different types of fluxons can form the stable bound states. Mixed bound states φ_1 and φ_2 are shown in Figs.5,6 for $h_e = 0.6$, $a_1 = 1$, $a_2 = -1$, $2l = 10$, $\gamma = 0$. They are characterized by $(N[\varphi_1] + N[\varphi_2])/2 = 1$ with fractional quantities $N[\varphi_1]$ and $N[\varphi_2]$. Stability of the mixed states depends on a_2 and h_e. For the large a_2 these solutions stabilize at sufficiently small h_e.

3. Summary
We show that the second harmonic contribution inspires an increasing of a complexity of coexisting static distributions in the LJJs described by the 2SG equation. Nonzero a_2 gives a rise new (stable and unstable) static distributions and produces the CBC-type states. We expect, the numerically predicted effects can be confirmed in experimental observations.

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