Numerical Study of Magnetic Flux in the LJJ Model with Double Sine-Gordon Equation

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Abstract. The decrease of the barrier transparency in superconductor-insulator-superconductor (SIS) Josephson junctions leads to the deviations of the current-phase relation from the sinusoidal form. The sign of second harmonics is important for many applications, in particular in junctions with a more complex structure like SNINS or SFIFS, where N is a normal metal and F is a weak metallic ferromagnet. In our work we study the static magnetic flux distributions in long Josephson junctions taking into account the higher harmonics in the Fourier-decomposition of the Josephson current. Stability analysis is based on numerical solution of a spectral Sturm-Liouville problem formulated for each distribution. In this approach the nullification of the minimal eigenvalue of this problem indicates a bifurcation point in one of parameters. At each step of numerical continuation in parameters of the model, the corresponding nonlinear boundary problem is solved on the basis of the continuous analog of Newton's method. The solutions which do not exist in the traditional model have been found. The influence of second harmonic on stability of magnetic flux distributions for main solutions is investigated.

Keywords: long Josephson junction, in-line geometry, Sturm-Liouville, double sine-Gordon, bifurcation, continuous analog of Newton's method, fluxon, Numerov's finite-difference approximation.

1 Introduction

Physical properties of magnetic flux in Josephson junctions (JJs) deserve the base of the modern superconducting electronics. Tunnel SIS JJs are known to be having the sinusoidal current phase relation. However, the decrease of the barrier transparency in the SIS JJs leads the deviations of the current-phase relation from the sinusoidal form [1]. We study the static magnetic flux distributions in the long JJs taking into account the second harmonic in the Fourier-decomposition of the Josephson current. The sign of the second harmonic depends on physical applications under consideration. It is important, in particular, in junctions like SNINS and SFIFS, where N is a normal metal and F is a weak metallic ferromagnet [2]. Interesting properties of long Josephson junctions with an arbitrarily strong amplitude of second harmonic in current phase relation were considered in [3].

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Our purpose was to investigate the effect of the second harmonic on the existence and stability of the magnetic flux distributions. Below, the numerical scheme and results of our stability analysis are demonstrated.

2 Mathematical Statement of the Problem

For a sufficiently wide class of JJ the superconducting Josephson current as a function of magnetic flux φ (phase difference of superconductors wave functions) can be represented as a sine series [4]:

$$I_S = I_c \sin\varphi + \sum_{m=2}^{\infty} I_m \sin m\varphi \,. \tag{1}$$

Using only first two terms of this expansion one can show [5] that the distribution of the magnitude $\varphi(x)$ along x-axis of the junction in the static regime [4] satisfies the double sine-Gordon equation (2SG).

$$-\varphi'' + a_1 \sin\varphi + a_2 \sin 2\varphi - \gamma = 0 \,, x \in (-l; l) \,. \tag{2}$$

Here and below the prime means a derivative with respect to the coordinate x. The magnitude γ is the external current, l is the semilength of the junction, a_1 and a_2 are parameters corresponding to I_c and I_2 in (1) respectively. They depend on the preparation technology of junctions [1,6]. All the magnitudes are dimensionless.

In the case of in-line geometry of the junction the boundary conditions for (2) have the form

$$\varphi'(\pm l) = h_e, \tag{3}$$

where h_e is external magnetic field.

From the mathematical viewpoint the transfer of the junction into dynamical regime [4] means [7,8] a stability loss (bifurcation) of all static solutions $\varphi(x)$ of (2), (3) at the parameters γ or h_e variation. Our stability analysis of $\varphi(x, p)$ was based on numerical solution of the corresponding Sturm-Liouville problem

$$-\psi'' + q(x)\psi = \lambda\psi, \quad \psi'(\pm l) = 0 \tag{4}$$

with a potential $q(x) = a_1 \cos\varphi + 2a_2 \cos 2\varphi$.

The minimal eigenvalue $\lambda_0(p) > 0$ corresponds to a stable solution. In case $\lambda_0(p) < 0$ solution $\varphi(x, p)$ is unstable. The case $\lambda_0(p) = 0$ indicates the bifurcation with respect to one of parameters $p = (l, a_1, a_2, h_e, \gamma)$.

3 Numerical Method

Numerical solving of the boundary problem (2),(3) was performed on the basis of the Continuous analog of Newton's method [8]. At each Newtonian iteration the corresponding linearized problem was solved using three-point Numerov's finite-difference approximation of the fourth order accuracy [9]. The discretization of the Sturm-Liouville problem (4) was realized with the help of standard second order finite-difference formulae. The calculation of the first several eigenvalues of the corresponding algebraic 3-diagonal problem was performed applying the standard subroutine from the package EISPACK. Details of numerical scheme are described in [10].

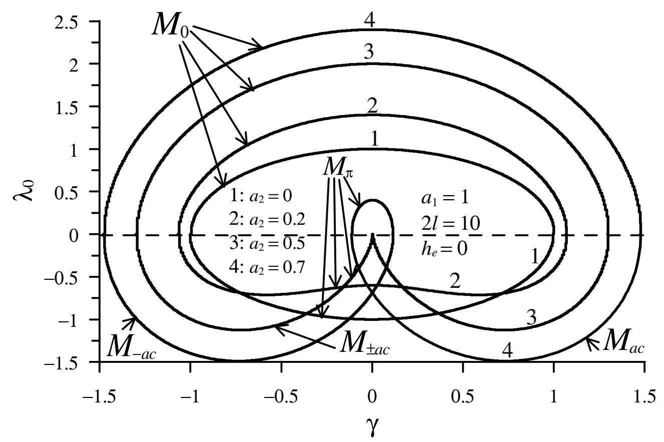

Fig. 1. Change of $\lambda_0(\gamma)$ for CS with increase of the coefficient a_2 in the interval $a_2 \in [0; 0.7]$ at $h_e = 0$, $a_1 = 1$, $2l = 10$

4 Numerical Results and Conclusions

Let us start with the *trivial solutions* of (2). In the "traditional" case $a_2 = 0$ two trivial solutions $\varphi = 0$ and $\varphi = \pi$ (below they are denoted by M_0 and M_π respectively) are known at $\gamma = 0$ and $h_e = 0$. Accounting of the second harmonic $a_2 \sin 2\varphi$ leads to appearing of two additional solutions $\varphi = \pm \arccos(-a_1/2a_2)$ (denoted as $M_{\pm ac}$). The corresponding λ_0 as functions of 2SG-equation coefficients have the form $\lambda_0[M_0] = a_1 + 2a_2$, $\lambda_0[M_\pi] = -a_1 + 2a_2$ and $\lambda_0[M_{\pm ac}] = (a_1^2 - 4a_2^2)/2a_2$. The exponential stability of these constant solutions (CS) is determined by the signs of the parameters a_1, a_2, and the ratio a_1/a_2 [10].

The dependencies of λ_0 on the external current γ for CS at several positive values of a_2 are demonstrated in Fig. 1. Arising of the stable states M_π by the external current γ at $a_2 > 0.5$ is shown.

When $a_2 < -0.5$ the stable solution M_0 disappears and other stable constant solutions $M_{\pm ac}$ arise. This transition is seen in Fig. 2.

In addition to CS, the 2SG equation admits *fluxon solutions*. The fluxons play a significant role in the JJ physics. Different distributions of magnetic flux in JJ are considered in the review [8]. At small external fields h_e such distributions are fluxon Φ^1, antifluxon Φ^{-1} and their bound states $\Phi^1\Phi^{-1}$ and $\Phi^{-1}\Phi^1$. As external

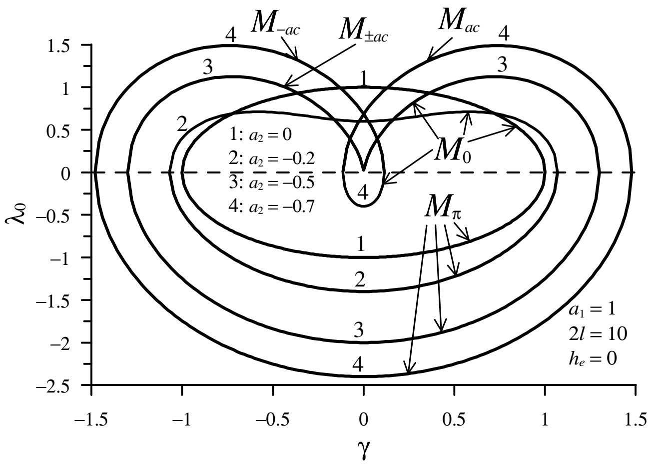

Fig. 2. The same as on Fig. 1 but for $a_2 \in [-0.7; 0]$

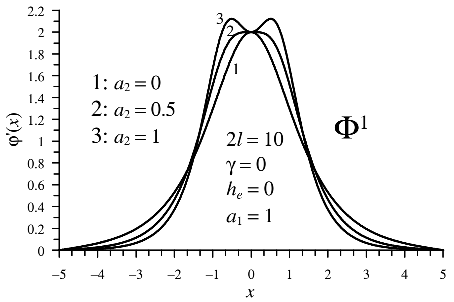

Fig. 3. Distribution of internal magnetic field of the fluxon Φ^1 for positive parameter a_2 at $\gamma = 0$, $h_e = 0$ and $2l = 10$

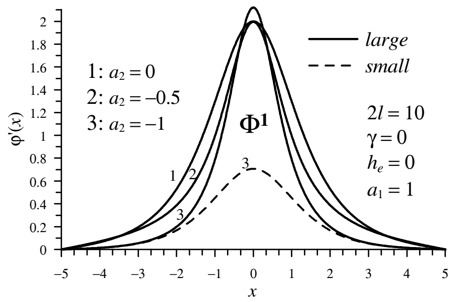

Fig. 4. The same as in Fig. 3 for negative a_2. The dashed line shows "small" solution.

magnetic field h_e is growing, more complicated stable fluxon and bound states appear: $\Phi^{\pm n}$ and $\Phi^{\pm n}\Phi^{\mp n}$ $(n = 1, 2, 3, \ldots)$.

Let us compare some basic physical characteristics of one-fluxon solution Φ^1 in our model (2)–(3) with the traditional one ($a_1 = 1$, $a_2 = 0$). In both models the value of the magnetic flux $\varphi(x)$ in the middle of junction is $\varphi(0) = \pi$. In Fig. 3 the deformation of the $\varphi'(x)$ under influence of the parameter $a_2 \in [0; 1]$ is demonstrated. At $a_2 = 0.5$ the curve of internal magnetic field $\varphi'(x)$ has a plateau in a neighborhood of the center of junction $x = 0$. Further increase of the parameter a_2 leads to a formation of two maxima of the magnetic field. Thus, the inclusion of the second harmonic leads to the qualitative change of fluxon distribution Φ^1. Such deformation does not appear with a decrease in parameter a_2 at $h_e = 0$ (Fig. 4). But, we observe a creation of new vortex when $a_2 < -0.5$ in zero magnetic field in agreement with the analytical results (see [3]

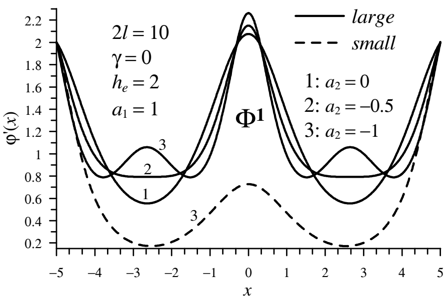

Fig. 5. The same as in Fig. 4 for $h_e = 2$

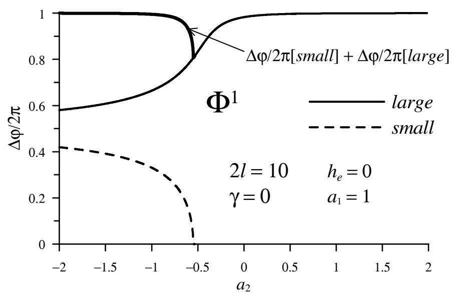

Fig. 6. Full magnetic flux for Φ^1 vs parameter $a_2 \in [-1; 1]$ at $h_e = 0$, $\gamma = 0$, $2l = 10$

and references there). This vortex is called as a "small" fluxon and coexisting fluxon solution as a "large" one. In cited work the new solution is investigated only at $h_e = 0$. In our work we show how the "small" fluxon is changed under the influence of the external magnetic field (Fig. 5). In the case of sufficiently large external magnetic field h_e a similar qualitative deformation is observed in the local minima regions only for the "large" fluxon when $a_2 < 0$ (see Fig. 5).

With change of the coefficient a_2 the number of fluxons [8]

$$N(p) = \frac{1}{2l\pi} \int_{-l}^{l} \varphi(x)\, dx,$$

corresponding to the "large" distribution Φ^1 is conserved, i.e., $\partial N/\partial a_2 = 0$. Here we have a value $N[\Phi^1] = 1$. But for the "small" vortex we have $N[small] = 0$, so in [11] we denote it as M_0.

At $a_2 > -0.5$ the full magnetic flux [8] $\Delta\varphi(p) = \varphi(l) - \varphi(-l)$ for "large" fluxon solution tends to 2π when a_2 is growing. As we can see in Fig. 6, at $a_2 \lesssim -0.5$ $\Delta\varphi[large] + \Delta\varphi[small] \approx 2\pi$ except the region around the bifurcation value of the second harmonic $a_2 = -0.5$. So, due to this relation the creation of "large" and "small" fluxons might be considered as a one process. We consider that the creation of new solutions at $a_2 \lesssim -0.5$ and their relation with the traditional ones need a special investigation.

One-fluxon "large" state remains unstable in zero external magnetic field for all considered values of the parameter a_2. The change of its stability under the influence of the field h_e is presented in [12].

In conclusion, we stress that new solutions we found do not exist in the traditional case ($a_2 = 0$). In this paper we focused on the stability analysis of constant and one-fluxon solutions only at different values of the a_2. Investigation of another classes of solutions of 2GS-equation is a point of further research.

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