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Numerical Simulation of Critical Dependences for Symmetric Two-Layered Josephson Junctions

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Abstract—Partial critical dependences of the form current–magnetic field in a two-layered symmetric Josephson junction are modeled. A numerical experiment shows that, for the zero interaction coefficient between the layers of the junction, jumps of the critical currents corresponding to different distributions of the magnetic fluxes in the layers may appear on the critical curves. This fact allows a mathematical interpretation of the results of some recent experimental results for two-layered junctions as a consequence of discontinuities of partial critical curves.

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1. INTRODUCTION

Recently, the properties of multilayered (in particular, two-layered) Josephson junctions (JJs) have been studied by a number of researches (see, e.g., [1–11]). Such structures make it possible to state and study new physical effects that do not occur in one-layered JJs; they are also of considerable practical interest.

One of the most interesting experimental results discovered in recent years is the so-called current locking effect [4, 5]. The essence of this phenomenon is as follows: some parts of the partial critical curves of the form current–electromagnetic field coincide for two-layered JJs. Thus, jumps of the critical current corresponding to some fixed values of the external electromagnetic field are observed on the curves obtained experimentally.

In this paper, we describe a numerical investigation of critical curves in the framework of the model of JJs based on inductively coupled layers. Our work is based on the ideas of the theory of partial stability [12, 13]. The transitions of the junction layers from the superconducting to the resistive regime are mathematically interpreted as bifurcations of the static distributions of the magnetic flux under the change of parameters (the impressed magnetic field and the external current). In the framework of the conventional long one-dimensional JJ model, this approach was first used in classical paper [14]. Each static distribution in a particular layer is assigned a regular Sturm–Liouville problem whose eigenvalues make it possible to judge the partial stability or instability of the distribution. In the framework of this approach, the critical current of a distribution is the value of the external current for which the corresponding minimal eigenvalue vanishes for the fixed remaining parameters (in particular, for the given intensity of the external magnetic field). For a certain given field, the corresponding nonlinear boundary value problem in the model may have more than one static solution. Therefore, the critical current of the layer is the maximal critical current among the distributions in this layer, and the critical current of the structure as a whole is the minimal critical current among the critical currents of the layers. The partial critical dependences for each of the layers are constructed as the envelopes of the bifurcation curves of the particular distributions.

2. STATEMENT OF THE PROBLEM

In the framework of the long JJs model, which takes into account only the inductive interaction between the neighboring layers, the dynamics of the magnetic flux in a JJ of length $2l$ is described (see [2]) by the system of perturbed sine-Gordon equations:

$$\phi_{tt} + \alpha\phi_t - A\phi_{xx} + J_z(\phi) + \Gamma = 0, \quad t > 0, \quad x \in (-l, l), \tag{1a}$$

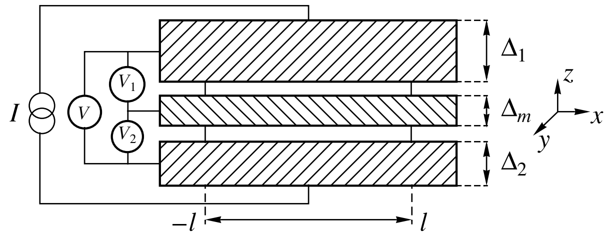

Fig. 1.

$$\phi_x(t, \pm l) = H. \tag{1b}$$

Here, t and x are the time and the space coordinate along the junction, respectively. All the magnitudes here and in what follows are dimensionless. Methods for reducing the equations to the dimensionless form are described, for example, in [2, 15]. The subscripts denote the derivatives with respect to the corresponding variable. $\phi(t, x) = (\phi_1, \phi_2)^T$ denotes the H-vector of magnetic fluxes in the lower $\phi_1(t, x)$ and the upper $\phi_2(t, x)$ layer, respectively. The constant $\alpha \geq 0$ is the dissipation coefficient. Below, we restrict the consideration to the case of symmetric two-layered JJs with identical layers; in particular, the thickness of these layers is identical: $\Delta_1 = \Delta_m = \Delta_2$ (see Fig. 1). Then, the vector of densities of the Josephson current is determined as $J_z(\phi) = (\sin\phi_1, \sin\phi_2)^T$. The vector of the external current $\Gamma = \gamma(1, 1)^T$; for simplicity, we assume that $\gamma = \text{const}$ below. The superscript T denotes the transposition. Similarly, we have $H = h_B(1, 1)^T$, where the constant h_B is the external magnetic field oriented along the axis y in the plane of the junction. The quadratic symmetric matrix $A(s)$ depends only on the coupling parameter $s \in (-1, 0]$:

$$A(s) = \frac{1}{1-s^2}\begin{pmatrix} 1 & -s \\ -s & 1 \end{pmatrix}.$$

An analytical expression for s in terms of hyperbolic functions can be found in [2].

The particular expressions for the initial conditions at $t = 0$ are not essential for the further considerations, and they are omitted here.

Note that Eq. (1) is a gradient-type (variational) equation. Since the boundary values at the points $x = \pm l$ do not explicitly depend on the time t, the total energy of the JJ (the Lyapunov functional)

$$E(t) = \int_{-l}^{l}\left\{\frac{1}{2}[\phi_t^2 + \langle\phi_x, A\phi_x\rangle] + U(\phi)\right\}dx - \langle AH, \Delta\phi\rangle$$

corresponding to a particular bounded solution $\phi(t, x)$ decreases with time:

$$\frac{dE}{dt} = -\alpha\int_{-l}^{l}\phi_t^2(t, x)dt \leq 0. \tag{2}$$

The angle brackets $\langle a, b\rangle$ denote the inner product of the vectors a and b. The density of the currents $U(\phi)$ is determined in terms of the gradient by $U'_\phi(\phi) = J_z(\phi) + \Gamma$, which yields $U(\phi) = 2 - \cos\phi_1 - \cos\phi_2 + \gamma(\phi_1 + \phi_2)$. The vector $\Delta\phi = (\Delta\phi_1, \Delta\phi_2)^T$ is the total magnetic flux:

$$\Delta\phi(t) = \phi(t, l) - \phi(t, -l). \tag{3}$$

It follows from (2) that any time-dependent solution $\phi(t, x)$ to Eq. (1) tends to a certain static solution as $t \longrightarrow \infty$; in particular, it tends to the static distribution $\varphi(x) \equiv (\varphi_1(x), \varphi_2(x))^T$.

From the physical point of view, the importance of studying the static solutions stems from the conditions of the experiment to measure the voltage–current characteristics of the layers and, in particular, from the dependences of the type *critical current–magnetic field* [6]. Assume that one or both layers operate in the resistive regime [15]. Then, the voltage measured at the layers $V_1(t) \neq 0$ and (or) $V_2(t) \neq 0$; moreover, by

Ohm's law, the total voltage $V(t)$ on the structure (see Fig. 1) is

$$V(t) = V_1(t) + V_2(t). \quad (4)$$

Here, the average voltages on particular layers are determined by the Josephson relations

$$V_i(t) = \frac{1}{2l}\int_{-l}^{l} \phi_{i,t}(t,x)dx, \quad i = 1, 2. \quad (5)$$

If both layers operate in the Josephson regime (i.e., for the given h_B, the current through the structure is less than a certain critical value), then the voltages measured on the layers are zero: $V_1(t) = 0$ and $V_2(t) = 0$; this yields $\phi_{i,t}(t, x) = 0$.

Note that, in the case of the capacitance interaction between the layers (see, e.g., [11] and references therein), the generalized Josephson relations must be used instead of (5).

Usually, Eq. (1) has more than one static solution. Therefore, in order to find the ultimate state of the junction, the stability of the static solutions must be analyzed. In the overlap geometry (see [15]), the vector (pair) of static distributions of the magnetic flux $\varphi(x) = (\varphi_1(x), \varphi_2(x))^{T}$ in a JJ satisfies the boundary value problem

$$-A\varphi_{xx} + J_z(\varphi) + \Gamma = 0, \quad (6a)$$

$$\varphi_x(\pm l) = H. \quad (6б)$$

Note that the solutions of problem (6) smoothly depend both on the space coordinate x and on the parameters l, s, h_B, and γ; i.e., $\varphi = \varphi(x, p)$, where $p \equiv \{l, s, h_B, \gamma\}$ ($p \in \mathcal{P} \subset \mathbb{R}^4$) is the 4-vector of the parameters of the model. Below, the particular form of the dependence on p is specified only when it is needed.

For clarity, we briefly consider some main types of static solutions in linear one-layered JJs. When the coupling coefficient $s \longrightarrow 0$, two-layered JJs have similar solutions.

In the zero field ($h_B = 0$) and under the zero current ($\gamma = 0$), each differential equation in (6) has many Meissner (trivial, vacuum) solutions of the form $\varphi(x) = k\pi$, where $k = 0, \pm1, \pm2, \ldots$.. In particular, the solution $\varphi(x) = 0$ is stable, and the solution $\varphi(x) = \pi$ is unstable. In what follows, the stable Meissner solutions are denoted by M.

The solutions that physically correspond to vortex distributions of the magnetic flux in the junction are more complex. The simplest vortex solutions are the fluxon (antifluxon) solutions (below, we use the notation $\Phi \equiv \Phi^1$ and Φ^{-1}), for which there are exact analytical expressions for the "infinite" junction ($l \longrightarrow \infty$), $h_B = 0$, and $\gamma = 0$:

$$\Phi_\infty^{\pm1} \equiv \varphi(x) = 4\arctan\exp(\pm x) + 2k\pi. \quad (7)$$

For a junction with a finite length, the objects of type (7) are not fluxons in the strict sense; however, some of their properties (in particular, their finite energy and size) justify the use of the same terminology.

For $|h_B| \ge 0$ and $\gamma = 0$, multufluxon distributions may exist in a one-layered junction of the finite length $2l$. For these distributions, we use the notation Φ^n, where the integer $n = \pm1, \pm2, \ldots$ is the number of vortices defined as the functional (see [16])

$$n \equiv N[\varphi](p) = \frac{1}{2l\pi}\int_{-l}^{l} \varphi(x, p)dx. \quad (8)$$

For an "infinite" junction, the expression on the right-hand side is interpreted as the limit.

Since any solution $\varphi(x)$ of problem (6) is determined up to $2k\pi$, the number $N[\varphi]$ is also determined up to $2k$. The arbitrariness in the choice of k can be used to "conform" the sign of n to the sign of the total magnetic flux of distribution (3), i.e., to the direction of the magnetic field of a vortex. In particular, for the fluxon solution Φ of form (7), we can set $k = 0$, and, for the antifluxon solution Φ_1, we set $k = -1$; then, it is easy to verify that $N[\Phi_\infty^{\pm1}] = \pm1$.

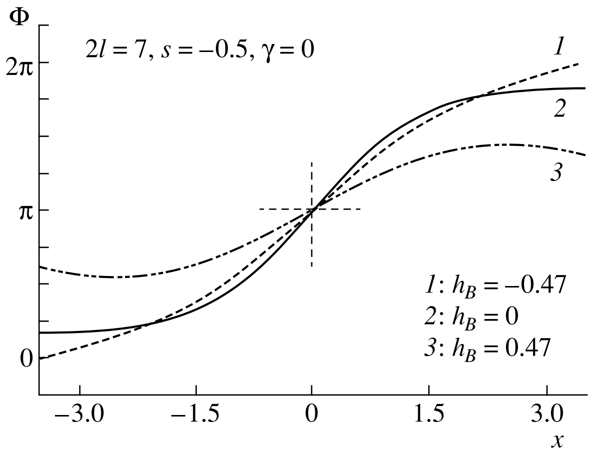

Fig. 2.

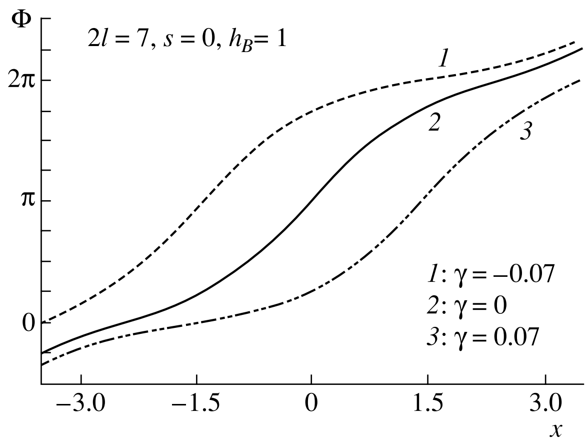

Fig. 3.

For time-dependent solutions $\phi_i(t, x)$ of nonstationary problem (1), the function $N_i[\phi_i](t)$ and the average voltage on the layer $\bar{V}_i(t)$ are related, due to (4), by the obvious equation

$$\bar{V}_i(t) = \pi N_{i,t}[\phi_i](t).$$

Therefore, the independence of the number of fluxons in the particular layer i of the time t (i.e., $N_{i,t} = 0$) is equivalent to the condition that the average voltage on the layer is zero. In the framework of the linear stability theory, the nonzero voltage on a layer is caused by a bifurcation of a stable distribution of the magnetic field.

By way of example, note that, for the external current $\gamma = 0$ and any admissible field h_B, the M-solution and any n-fluxon (antifluxon) distribution Φ^n are characterized by the values $N[M] = 0$ and $N[\Phi] = \pm n$, respectively. In other words, functional (8) is independent of the parameter h_B. Figure 2 shows Φ-curves for $\gamma = 0$ and for three values of the external field h_B. In this case, the geometric interpretation of the preservation of $N[\Phi]$ is that the areas under the curves corresponding to various values of h_B are preserved, and the height of the "center" $\varphi(0) = \pi$ is also independent of h_B. Note that $N[\Phi]$ is also independent of the field h_B for any distribution of the magnetic flux in all linear static JJ models in the absence of the external current γ or the absence of the current caused, for example, by a variable geometry of the junction (see [17]).

In the given field h_B, the external current $\gamma \neq 0$ pushes the magnetic field out of the JJ; more precisely, it shifts the magnetic field from the center $x = 0$ to the right ($\gamma > 0$) or to the left ($\gamma < 0$). This effect is illustrated in Fig. 3 for the Φ-distribution in the field $h_B = 1$ for three values of the parameter γ. For the distributions Φ^n and the current $\gamma \neq 0$, the number of vortices (8) becomes fractional.

For the current $\gamma = \gamma_{cr}$, an initial distribution that is stable for $|\gamma| < |\gamma_{cr}|$ breaks down and goes to an unstable state. The value γ_{cr} is called the critical current for the distribution. In a physical experiment, the loss of stability (the breakdown) of a distribution with the maximal critical current under the given h_B corresponds to the most likely transition of the JJ from the static state with the zero voltage to the quasi-particle branch of the voltage–current characteristic. It is clear that the realization of less probable (less energetically advantageous) distributions is possible in concrete physical experiments (see, e.g., [18]). In other words, the critical current of a junction is the maximal current for which the junction remains superconducting. From the mathematical viewpoint, the breakdown of a distribution indicates that a particular solution of problem (6) has a bifurcation when the parameter γ changes [14].

For certain values of the parameters h_B and γ, more complex vortex configurations can appear in a JJ; these configurations can be considered as a result of a nonlinear interaction of chains of fluxons and (or) antifluxons. Such solutions, which we will denote by $\Phi^n\Phi^m$ with $|n| + |m| > 0$, can have a fractional number of vortices (8); for a given value h_B, these solutions appear in pairs corresponding to the exchange of the degrees $n \Leftrightarrow m$. The sum of the numbers of vortices in the pair is an integer (see [16]). The simplest example is provided by the breather solutions of the form $B \equiv B^1$ and B^{-1} (see below) for which we have $N[B^{\pm 1}] = \pm 1$ for $\gamma = 0$.

In two-layered JJs with $s < 0$, the distributions of the magnetic flux in the individual layers are distorted due to the interaction between the layers and form coupled pairs. For the same values of the parameters h_B and γ, the distributions in the individual layers can be stable or unstable. In a physical experiment, the transition to the instability of a certain partial distribution when reaching the critical current $\gamma = \gamma_{cr}$ manifests itself in an abrupt appearance of a voltage on one of the layers and, therefore, due to the Ohm law (4), on the junction as a whole (see Fig. 1). In multilayered JJs, the situation is much more complex because one can perform experiments for determining the critical current for individual layers, for groups of layers, and for the structure as a whole.

3. STABILITY OF THE MAGNETIC FLUX DISTRIBUTIONS

3.1. Variational Principle

Equation (6a) and boundary conditions (6b) can be considered as necessary conditions for the extremum of the total energy of the junction

$$F[\varphi] = \int_{-l}^{l}\left[\frac{1}{2}\langle\varphi_x, A\varphi_x\rangle + U(\varphi)\right]dx - \langle AH, \Delta\varphi\rangle. \tag{9}$$

The domain of functional (9) is the set of all smooth vector functions $\varphi(x)$ on the interval $[-l, l]$ with the nodes belonging to the vertical lines $x = \pm l$. It is easy to verify that Eq. (6a) is the Euler–Lagrange equation for (9), and the boundary conditions (6b) are a consequence of the Weierstrass–Erdmann condition (see [19]).

For the further consideration, it is convenient to write the total energy (9) as the sum

$$F[\varphi] = F_1 + F_2 + F_{12}. \tag{10}$$

Here, F_i are the energies of the noninteracting layers:

$$F_i[\varphi_i] = \int_{-l}^{l}\left(\frac{1}{2}\varphi_{i,x}^2 + 1 - \cos\varphi_i + \gamma\varphi_i\right)dx - h_B\Delta\varphi_i.$$

The binding energy F_{12} is calculated by the formula

$$F_{12}[\varphi_1, \varphi_2] = \frac{s}{1-s^2}\int_{-l}^{l}\left[\frac{s}{2}(\varphi_{1,x}^2 + \varphi_{2,x}^2) - \varphi_{1,x}\varphi_{2,x}\right]dx + \frac{s}{1+s}h_B(\Delta\varphi_1 + \Delta\varphi_2).$$

From the theoretical point of view, the investigation of the stability of the distributions of the magnetic flux in a one-layered JJ is reduced (see [14]) to finding the first several eigenvalues of a Sturm–Liouville problem with the potential determined by a particular distribution. In the framework of this approach, the stability of a distribution is equivalent to the positive definiteness of the second variation of the total energy. An application of the same idea to a wide range of problems in mechanics and physics is considered in [20–23].

3.2. Partial Stability of Distributions in the Layers of a JJ

To analyze the stability of distributions of the magnetic flux in the ith layer of a two-layered JJ, it is required, according to the partial stability theory (i.e., the stability with respect to a part of the variables) [13], to consider the second variation of functional (10) with the fixed distribution in the neighboring layer. It is assumed that the pair $\varphi(x)$ is a solution of nonlinear boundary value problem (6). The investigation of the nature of the extremum of functional (10) at the point $\varphi(x)$ is equivalent to the investigation of the nature of the extremum of the sum $F_i + F_{12}$. For this purpose, consider the family of perturbations $\varphi_i(x) + \varepsilon u_i(x)$ of the state $\varphi_i(x)$ in the ith layer of the junction. Here ε is the parameter of the family. Hence, we obtain a regular Sturm–Liouville problem in the usual way:

$$-a_{ii}u_{i,xx} + q_i(x)u_i = \mu_i u_i, \tag{11a}$$

$$u_{i,x}(\pm l) = 0, \tag{11b}$$

$$\int_{-l}^{l} u_i^2(x)dx - 1 = 0. \tag{11c}$$

The potential of this problem $q_i(x) = \cos\varphi_i(x)$ is determined in terms of the solution $\varphi_i(x)$. Below, problem (11) is called the Sturm–Liouville problem for the ith layer.

On the finite interval $[-l, l]$, problem (11) has a discrete spectrum, which is bounded from below (see [19]):

$$-1 \le \mu_{i\,\min} \equiv \mu_{i0} < \dots \mu_{in} < .$$

Each eigenvalue μ_{in} corresponds to a unique (up to the sign) eigenfunction $u_{in}(x)$ satisfying normalization condition (11c). The function $u_{in}(x)$ has n roots on the interval $(-l, l)$. In particular, the eigenfunction $u_{i0}(x)$ corresponding to the minimal eigenvalue μ_{i0} has no roots on $(-l, l)$.

Since the potential $q_i(x)$ is determined in terms of the solutions of boundary value problem (6), the eigenvalues and the eigenfunctions of Sturm–Liouville problem (11) are also smooth functions of the parameters p; i.e., $\mu_{in} = \mu_{in}(p)$ and $u_{in} = u_{in}(x, p)$.

We say that there is a bifurcation of order $n \ge 0$ of the distribution $\varphi_i(x)$ if the nth eigenvalue of the corresponding Sturm–Liouville problem (11) satisfies the condition

$$\mu_{in}(p) = 0. \tag{12}$$

The bifurcation of the minimal order $n = 0$ corresponds to the bifurcation of the stable static distribution $\varphi_i(x)$ into an unstable distribution and conversely (see [21]). Equation (11a) is the Jacobi equation (see [24]) for the functional $F_i + F_{12}$. For the stable solutions, we have $\mu_{i0}(p) > 0$, and, therefore, the functional $\delta^2(F_i + F_{12})$ is positive definite.

Therefore, the spectrum of Sturm–Liouville problem (11) can be used to judge the stability of $\varphi_i(x)$ in the linear approximation. Thus, for $\mu_{i0} > 0$, we can call the solution $\varphi_i(x)$ partially stable. For $\mu_{i0} < 0$, the solution $\varphi_i(x)$ is said to be partially unstable. For $n > 0$, we have the bifurcation of a partially unstable solution of problem (6) into another partially unstable solution.

Below, we assume that the length of the junction $2l$ and the coupling parameter s are fixed (bifurcations of the vortices in a JJ with varying l were studied in [17, 24]). The equation

$$\mu_{i0}(h_B, \gamma) = 0 \tag{13}$$

implicitly determines the bifurcation curve for the distribution $\varphi_i(x)$ on the parametric plane $\mathcal{P}_{h\gamma} \equiv \{h_B, \gamma \in \mathbb{R}^2$. For the given field h_B, the value $\gamma = \gamma_{\rm cr}$ satisfying Eq. (13) is called the critical current for the corresponding distribution of the magnetic flux. The maximal critical current among the stable distributions that exist for the given h_B is the critical current of the layer. According to formula (4), the critical current of the two-layered JJ is the minimal of the critical currents of the layers.

Note that, in one-layered JJs, the bifurcations of order $n \ge 1$ are mainly of mathematical interest because the lifetime of unstable distributions is very small (see [14]). Below, we show that, in two-layered JJs, the partial high-order bifurcations of magnetic fluxes can play an important role due to the interaction between the layers.

Consider an important particular case. The solution $\varphi(x)$ to Eq. (6a) is called symmetric (see [6]), i.e., the pair $\varphi(x)$ is called symmetric, if $\varphi_1(x) = \varphi_2(x)$. Examples of symmetric pairs of solutions are given below. For symmetric distributions of the magnetic flux, boundary value problem (6) is considerably simpler (for convenience, we omit the index i):

$$-a\varphi_{xx} + \sin\varphi + \gamma = 0, \quad \varphi_x(\pm l) = h_B. \tag{14}$$

Here, the coefficient $a = a_{11} + a_{12} = 1/(1 + s)$, $a \in [1, \infty)$; moreover, $a \ge a_{ii}$.

3.3. Algorithms for Calculating Bifurcation Points

To find the bifurcation points of solutions of problem (6), we consider, as in [25], Eqs. (6) and (11) as a system of equations; the general solution of this system depends on five parameters p and μ_i. To find a partial solution, four of these parameters must be fixed, and the fifth one is found as the solution of the system.

Depending on which parameters are fixed, two classes of problems are considered.

In the simplest case, when all the coordinates of the vector p are defined, the problem is split into independent subsystems (6) and (11). The eigenvalue μ_i is to be found, which corresponds to the verification of the partial stability of the coordinate $\varphi_i(x)$ of the pair $\varphi(x)$, which is already known. Assume that, for a certain h_B (or for a certain γ), $\mu_i > 0$. Varying the field h_B under the fixed current γ (or varying γ under the fixed h_B), one can satisfy condition (13) up to any desired accuracy; i.e., one can find one point on the bifurcation curve corresponding to the solution $\varphi_i(x)$.

If μ_i is given, then system of equations (6), (11) is closed with respect to the unknown functions ($\varphi(x)$, $u_i(x)$) and one of the parameters from the set p, which is denoted by ξ (respectively, by $\check{p}$, we denote the set of the parameters p minus ξ). To solve this nonlinear eigenvalue problem with the spectral parameter ξ, it is convenient to use the iterative algorithm (see [17, 25]) based on the continuous analog of the Newton method (CANM) proposed in [26]. Defining $\mu_i = 0$, one can find an initial approximation to satisfy condition (12) for $n \geq 0$. Thus, we find the surface $\xi(\check{p}; \varphi)$ in the space $\mathcal{P}$ corresponding to the bifurcation of order n of the solution $\varphi_i(x)$. The cuts of this surface by the planes corresponding to two fixed parameters from the set $\check{p}$ are the bifurcation curves $\mu_{in}(\xi, \eta) = 0$ on the plane $\mathcal{P}_{\xi\eta}$ for the remaining parameter η from $\check{p}$ and the parameter ξ.

This method for the direct calculation of the bifurcation points of the magnetic flux distributions in one-dimensional long JJs was proposed in [25] and was used for solving a wide range of physical problems in [16].

Since boundary value problem (6) usually has more than one stable solution for the given h_B and γ, the partial critical curve for the ith layer of the junction is constructed as the envelope of the bifurcation curves of different solutions in the same layer. In particular, the partial critical curve of the form current–magnetic field consists of parts of similar curves (13) corresponding various vortices with the maximal (for the given h_B) current γ.

Now, we briefly consider the calculation of the intersection points of parts of the partial bifurcation curves for the ith layer. Let us fix the geometric parameters l and s. Let (h_c, γ_c) be the coordinates of an intersection point C on the plane $\mathcal{P}_{h\gamma}$. Each such point corresponds to vanishing of the eigenvalue of the Sturm–Liouville problem (11) for two solutions $\varphi^L(x)$ and $\varphi^R(x)$ whose bifurcation curves lie to the left (L) and to the right (R) of the point C. Denote by (μ_i^L, $u_i^L(x)$) and (μ_i^R, $u_i^R(x)$) the eigenvalues and the eigenfunctions of the partial Sturm–Liouville problems of form (11) corresponding to the distributions $\varphi_i^L(x)$ and $\varphi_i^R(x)$. Then, provided that the conditions

$$\mu_i^L(h_c, \gamma_c) = 0, \quad \mu_i^R(h_c, \gamma_c) = 0,$$

are satisfied, we arrive at the following boundary value problem:

$$-A\varphi_{xx}^L + J_z(\varphi^L) + \Gamma_c = 0, \quad \varphi_x^L(\pm l) = H_c, \quad -a_{ii}u_{i,xx}^L + \cos\varphi_i^L u_i^L = 0, \quad u_{i,x}^L(\pm l) = 0, \tag{15a}$$

$$-A\varphi_{xx}^R + J_z(\varphi^R) + \Gamma_c = 0, \quad \varphi_x^R(\pm l) = H_c, \quad -a_{ii}u_{i,xx}^R + \cos\varphi_i^R u_i^R = 0, \quad u_{i,x}^R(\pm l) = 0, \tag{15b}$$

where $H_c = h_c(1, 1)^{\mathrm{T}}$ and $\Gamma_c = \gamma_c(1, 1)^{\mathrm{T}}$. System (15) contains six unknown functions ($\varphi^L(x)$, $u_i^L(x)$) and ($\varphi^R(x)$, $u_i^R(x)$) and two unknown parameters—the field h_c and the current γ_c. To close this problem, we add two normalization conditions for the partial wave functions $u_i^L(x)$ and $u_i^R(x)$; for example,

$$\int_{-l}^{l} [u_i^L(x)]^2 dx = 1 \quad \text{and} \quad \int_{-l}^{l} [u_i^R(x)]^2 dx = 1. \tag{16}$$

To solve the two-parameter nonlinear eigenvalue problem (15), (16), one can use a CANM-based algorithm with appropriate initial approximations.

4. NUMERICAL METHODS

To solve the nonlinear boundary value problems considered above, we used an iterative CANM-based algorithm. The linearized boundary value problems that appear at each iteration step were numerically solved using the Galerkin finite element method [27].

A discretization based on the finite element method was also used to construct a difference scheme for the Sturm–Liouville problem that arises in checking the stability of the solutions. The corresponding algebraic eigenvalue problems for matrix pencils were solved by the subspace iteration method [28].

By way of example, consider the construction of a difference scheme for the Sturm–Liouville problem. Rewrite Eq. (11) in the form

$$-au_{xx} + q(x)u = \mu u, \tag{17a}$$

$$u_x(\pm l) = 0, \tag{17b}$$

$$\int_{-l}^{l} u^2(x)dx - 1 = 0. \tag{17c}$$

Take the scalar product of Eq. (17a) and the trial function $v(x)$:

$$v(x) \in H^1(-l, l) = \{v : v, v_x \in L^2(-l, l)\},$$

and integrate by parts using conditions (17b):

$$\int_{-l}^{l} [au_x v_x + q(x)uv]dx = \mu \int_{-l}^{l} uv(x)dx \quad \forall v \in H^1.$$

Here and in what follows L^2 and H^1 are the well-known functional spaces (see [29]).

For the discretization of the integral identity, we use isoparametric quadratic finite elements.

Let $\Pi = \{-l = x_1 < \ldots < x_n = l\}$, where $x_{i+1} - x_i = h_i \le h$ is a partitioning of the interval $[-l, l]$ into the finite elements $e_i = [x_i, x_{i+1}]$, $i = 1, 2, \ldots, N-1$.

Furthermore, let $S^h = \{v^h(x), v^h(x) \in C[-l, l], v^h(x)\big|_{e_i} \in P_2\}$ be the finite-dimensional set of trial functions ($S^h \subset H^1$) and P_2 be the set of quadratic polynomials. In the space $\mathbb{R} \times S^h$, we seek a pair $\{\mu^h, u^h\}$ such that

$$\int_{-l}^{l} [au_x^h v_x^h + q(x)u^h v^h]dx = \mu^h \int_{-l}^{l} u^h v^h dx \quad \forall v^h \in S^h.$$

Transform the element $e_i = [x_i, x_{i+1}]$ into the standard element $E = [-1;1]$ using the transformation $x = x_i + (\theta + 1)(x_{i+1} - x_i)/2$. Here, the parameter $\theta \in [-1, 1]$.

Denote by $\psi(\theta) = (\psi_1(\theta), \psi_2(\theta), \psi_3(\theta))^{\mathrm{T}}$ the vector of quadratic shape functions for the standard element E with the components

$$\psi_1(\theta) = \theta(\theta - 1)/2, \quad \psi_2(\theta) = 1 - \theta^2, \quad \psi_3(\theta) = \theta(\theta + 1)/2.$$

Then,

$$v_{e_i}^h(x(\theta)) = \psi(\theta)v_{e_i}, \quad u_{e_i}^h(x(\theta)) = \psi(\theta)u_{e_i},$$

$$v_{x, e_i}^h(x(\theta)) = \psi'(\theta)v_{e_i}, \quad u_{x, e_i}^h(x(\theta)) = \psi'(\theta)u_{e_i},$$

where $v_{e_i} = (v_1, v_2, v_3)^{\mathrm{T}}$ and $u_{e_i} = (u_1, u_2, u_3)^{\mathrm{T}}$ are the 3-dimensional vectors of the values of the functions u^h and v^h at the nodes of the element e_i and the prime denotes the differentiation with respect to the local variable θ.

In this notation, we have, for the left-hand side of the integral identity,

$$\int_{-l}^{l} [au_x^h v_x^h + q(x)u^h v^h]dx$$

$$= \sum_{e_i} \int_{e_i} [a(v^h_{x, e_i})^{\mathrm{T}} (u^h_{x, e_i}) + q(x)(v^h_{e_i})^{\mathrm{T}} (u^h_{e_i})] dx$$

$$= \sum_{e_i} \left[v^{\mathrm{T}}_{e_i} |J|^{-1} a \int_{-1}^{1} [\psi']^{\mathrm{T}} [\psi'] ds u_{e_i} + v^{\mathrm{T}}_{e_i} |J| \int_{-1}^{1} q(x(\theta)) \psi^{\mathrm{T}} \psi ds u_{e_i} \right]$$

$$= \sum_{e_i} [v^{\mathrm{T}}_{e_i} K^1_{e_i} u_{e_i} + v^{\mathrm{T}}_{e_i} K^0_{e_i} u_{e_i}] = \sum_{e_i} [v^{\mathrm{T}}_{e_i} (K^1_{e_i} + K^0_{e_i}) u_{e_i}]$$

$$= \sum_{e_i} v^{\mathrm{T}}_{e_i} K_{e_i} u_{e_i} = V^{\mathrm{T}} K U^h,$$

where $|J| = (x_{i+1} - x_i)/2$ is the Jacobian of the transformation, $K = K^1 + K^0 = [k_{ij}, i, j = 1, 2, \dots, 2N - 1]$ is the global stiffness matrix [28], and $U^h = (U_1, \dots, U_{2N-1})^{\mathrm{T}}$ is the global vector of the values of the function $u^h(x)$ at the points of the finite element grid.

Similarly, for the right-hand side of the integral identity, we have

$$\int_{-l}^{l} u^h v^h dx = \sum_{e_i} \int_{e_i} (v^h_{e_i})^{\mathrm{T}} (u^h_{e_i}) dx$$

$$= \sum_{e_i} v^{\mathrm{T}}_{e_i} |J| \int_{-1}^{1} \psi^{\mathrm{T}} \psi ds u_{e_i}$$

$$= \sum_{e_i} v^{\mathrm{T}}_{e_i} M_{e_i} u_{e_i} = V^{\mathrm{T}} M U^h,$$

where M is the global matrix of mass (see [28]).

As a result, we obtain the algebraic eigenvalue problem

$$KU^h = \mu^h M U^h \tag{18}$$

for the matrix pencil with the symmetric matrices K and M.

For the numerical implementation, it is sufficient to store and use only the upper profiles of the matrices in the form of a *sky line* (see [28]).

Problem (18) is solved using the subspace iteration method [28].

5. NUMERICAL EXPERIMENT

The numerical experiment was performed for the JJ of length $2l = 7$ for two values of the coupling parameter $s = -0.3$ and $s = -0.5$.

The critical curves were constructed by varying a "bifurcation" solution of problem (6) with respect to one of the parameters h_B or γ while preserving, for example, $\mu_1 = 10^{-3}$. For this purpose, we used the algorithms of parametric continuation similar to those presented in [30].

First, we note that if the pair $\varphi = (\varphi_1, \varphi_2)^{\mathrm{T}}$ is a solution of boundary value problem (6), then the pair $\bar{\varphi} = (\varphi_2, \varphi_1)^{\mathrm{T}}$ is also a solution of (6). For symmetric solutions, $\varphi = \bar{\varphi}$.

Examples of pairs of distributions in the two-layered JJ with $s = -0.5$ and the influence of the field h_B on the stability of those pairs are given in Figs. 4–9, 13. In particular, Fig. 4 shows the plots of $\varphi_x(x)$ for the stable Meissner solution M in a small neighborhood ($\mu_{10} = \mu_{20} = 10^{-3}$) of two bifurcation points: ($h_B = h_{c1} = 1.2$, $\gamma_{\mathrm{cr}} \approx 0.217$) and ($h_B = h_{c2} \approx 2.03$, $\gamma_{\mathrm{cr}} = 0$). The solution M is symmetric; i.e., $\varphi_1(x) = \varphi_2(x)$. The magnetic self-field (the derivatives $\varphi_x(x)$) of such distributions is confined in a neighborhood of the layers' boundaries.

From the physical point of view, this is in agreement with the Meissner effect of pushing out the external magnetic field (screening) by the junction (see [15]).

Figure 5 shows the "bifurcation" plots of $\varphi_x(x)$ for the one-fluxon distribution Φ for $\gamma_{cr} = 0$ and $\mu_{10} = \mu_{20} = 10^{-3}$; i.e., on the boundaries $h_B = h_{c1} \approx 1.17$ and $h_B = h_{c2} \approx 2.03$ of the region where the fluxon with respect to the field h_B exists. In this case, the magnetic self-fields $\varphi_{i,x}(x)$ are also symmetric with respect to the ordinate in a neighborhood of the center $x = 0$ of the JJ.

The plots of the magnetic self-fields $\varphi_{i,x}(x)$ for the asymmetric pairs of the form (Φ, Φ^{-1}) for $h_B = 0$ and $h_B = 1$ in the absence of the external current γ are presented in Fig. 6. The solid curves correspond to the solitons in the first (lower) layer, and the dashed curves correspond to the solitons in the second (upper) layer. Note that the partial stability deteriorates when the boundary field h_B increases. This is due to the redistribution of the magnetic self-fields in the layers. Moreover, $N[\varphi_1] = 1$ and $N[\varphi_2] = -1$ for all admissible h_B.

Figure 7 shows the plots of the functions $\varphi_x(x)$ for the asymmetric pairs of the form (Φ, M) for $h_B = 0$, $h_B = 1$, and the current $\gamma = 0$. For $h_B = 0$, the deviation of the stable M-component ($\Delta\varphi_2 \approx 0.023$) from the trivial solution $\varphi_2(x) = 0$ ($\Delta\varphi_2 = 0$) for $s = 0$ is caused by the interaction between the layers. For the current $\gamma = 0$, functional (8) retains the constant values $N[\varphi_1] = 1$ and $N[\varphi_2] = 0$ on the solutions of this form in the entire domain of variation of the field h_B.

The distributions shown in Fig. 8 should be considered as a result of the nonlinear interaction of the unstable Meissner solution $\overline{M}$ ($\varphi_1(x) = \pi$ for $s = 0$, $h_B = 0$, and $\gamma = 0$) with the fluxon Φ; i.e., these are the pairs of the form $(\overline{M}, \Phi)$. For such solutions, we have $N[\varphi_1] = 1$ and $N[\varphi_2] = 1$ for $\gamma = 0$ in the domain of the admissible fields h_B.

The distributions of the magnetic self-fields (B, B^{-1}) for $h_B = 1.4$ are shown in Fig. 9. The curves *1* correspond to the case $\gamma = 0$, when the solutions have the symmetry $\varphi^1(x) = -\varphi^2(-x) + 2\pi k$; i.e., their derivatives are symmetric with respect to the vertical line $x = 0$ (here, the superscript indicates the number of the pair). In addition to curves *1*, there is the conjugate pair (B^{-1}, B). On such solutions, functional (8) takes fractional values; however, the sum $N[\varphi_i^1] + N[\varphi_i^2] = 2$. Curves *2* are calculated for the critical current $\gamma = \gamma_{cr} \approx 0.14$.

Let us proceed to the construction of the partial critical curves current–magnetic field in the two-layered JJ.

In the case of a symmetric matrix $A(s)$, the partial bifurcation dependences for the first and the second layers of the JJ are identical. For this reason, we will consider only the partial dependences for the distributions in the first layer.

Assume that, for a fixed γ and a certain h_B, the function $\varphi(x, h_B)$ is a solution of problem (6). Then, the function $2\pi k - \varphi(x, -h_B)$ ($k = 0, \pm1, \pm2, \ldots$) is also a solution in the field $-h_B$.

For the JJs with overlap geometry, which are considered in this paper, the current γ does not appear in the boundary conditions. Then, if $\varphi(x, \gamma)$ is a solution for a certain γ and if the sign of the current changes (i.e., the sign of the last term in (6a) changes), we have the solution $\varphi(x, -\gamma)$.

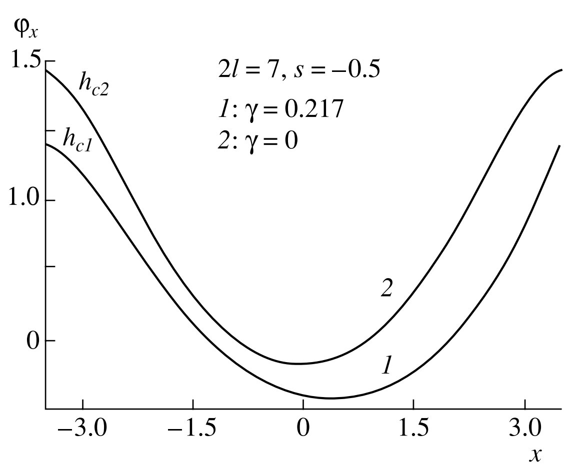

Fig. 4.

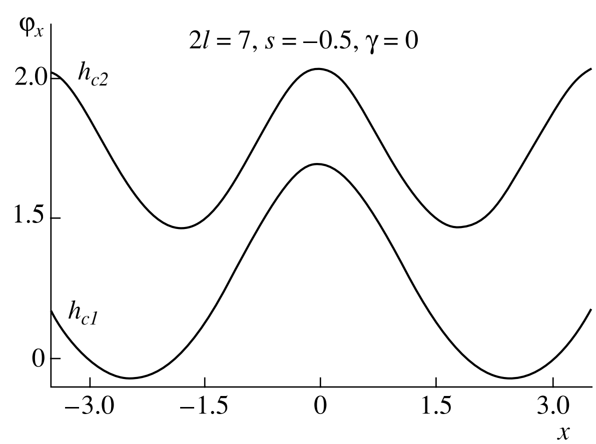

Fig. 5.

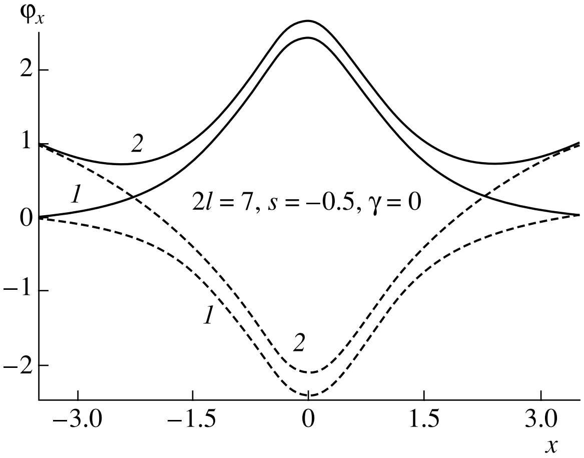

Fig. 6.

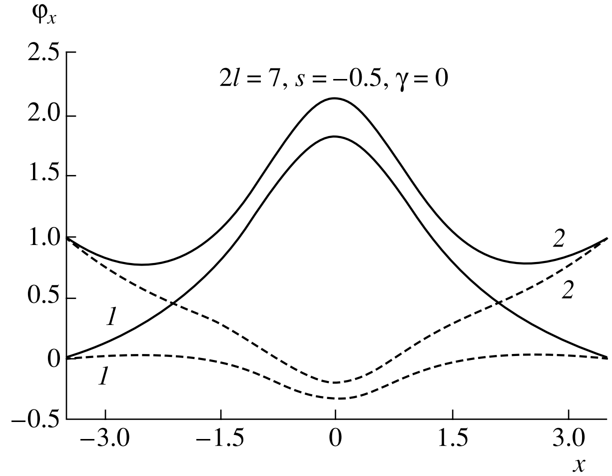

Fig. 7.

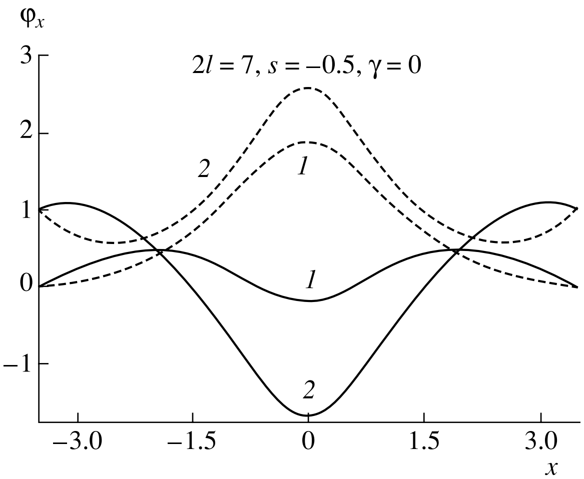

Fig. 8.

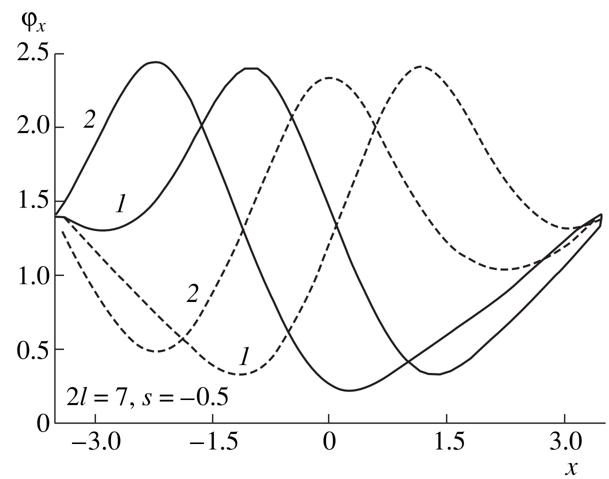

Fig. 9.

This fact implies the symmetry of the critical dependences with respect to the change of the direction of the axes $\gamma \Leftrightarrow -\gamma$ and $h_B \Leftrightarrow -h_B$ on the plane $\mathcal{P}_{h\gamma}$. These symmetries make it possible to restrict the considerations to the first quadrant.

Figure 10 shows some bifurcation curves of form (13) for some distributions $\varphi_1(x)$ of the magnetic flux in the first layer of a JJ. Note that different solutions are associated with different signs of the partial $\mu_{20}(h_B, \gamma)$ of the eigenvalues.

A more detailed description of the solutions corresponding to particular bifurcation curves is impossible in this paper because they are too numerous. For this reason, we restrict ourselves to a description of the most important (for the critical curve) solutions that have, for the given magnetic field h_B, the maximal critical current γ_{cr}. Parts of bifurcation curves (13) of such solutions form the critical curve (see Fig. 11) of the first layer in the JJ. The points of the continuous "joint" of individual parts are solutions of the nonlinear two-parameter eigenvalue problem (15), (16). However, numerical experiments demonstrate that the partial critical curve can have points of discontinuity, which are denoted by vertical dashed lines in Fig. 11 (see also Fig. 12). The values of the external magnetic field h_B for which there are discontinuities and the values of the jumps of the critical current $\Delta\gamma_{cr} \equiv \gamma_{cr}(S_R) - \gamma_{cr}(S_L)$ strongly depend on the particular model of the junction and on the parameters s and l. Here, $\gamma_{cr}(S_R)$ and $\gamma_{cr}(S_L)$ are the critical currents of the solutions to the right (R) and to the left (L) of the point of discontinuity.

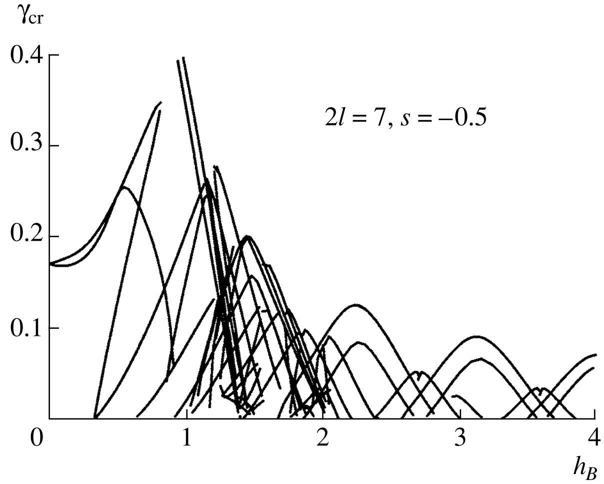

Fig. 10.

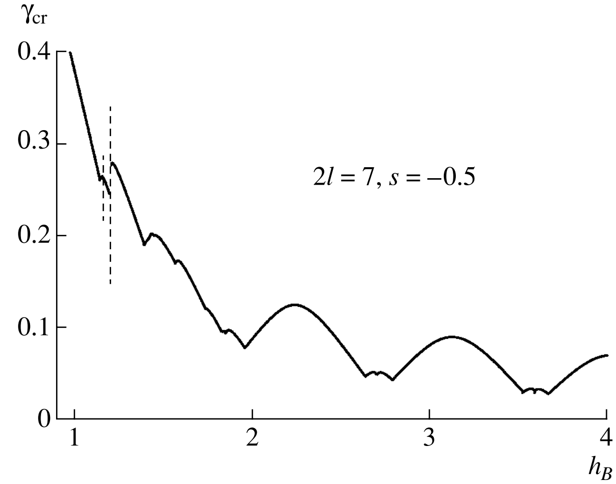

Fig. 11.

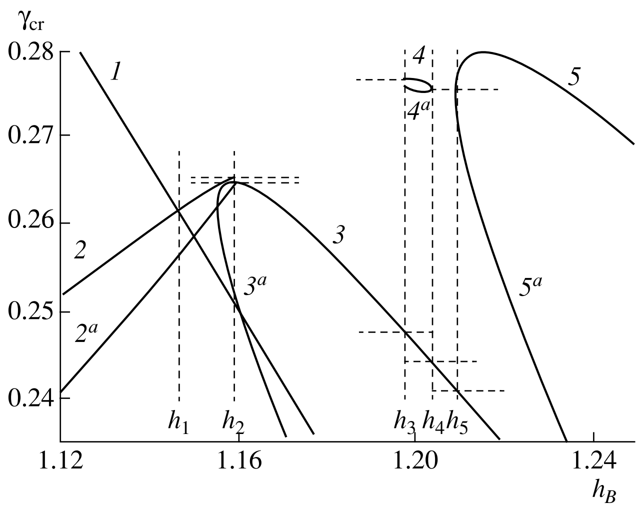

Fig. 12.

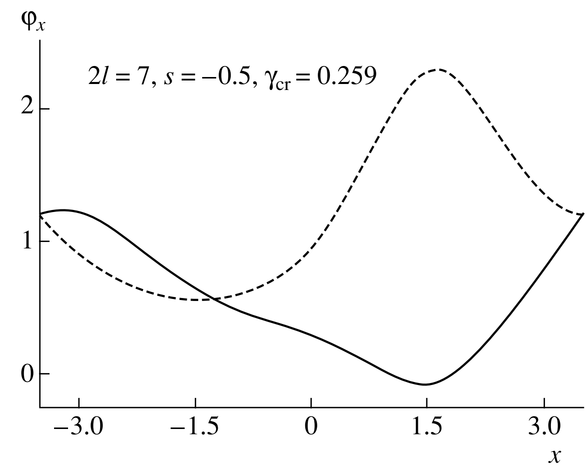

Fig. 13.

The mechanisms of forming such points are demonstrated in Fig. 12. For $h_B < h_1$, the symmetric Meissner pair 1 makes the dominating contribution (see Fig. 4) because, in this part, this pair has the maximal critical current compared to all other solutions of problem (6). For $h_B = h_1$, the bifurcation curve of the Meissner solution intersects the bifurcation curve of pair 2 of the form (Φ, M) (see Fig. 7). At the intersection point, pairs 1 and 2 are solutions of problem (15), (16). Therefore, on the interval $h_B \in [h_1, h_2)$, the critical curve of the JJ is formed by a part of bifurcation curve (13), which corresponds to pair 2.

For $h_B = h_2$, the critical curve goes, in a jump, to the branch generated by pair 3 of the form (M, Φ) (this solution for $h_B = 1.2$ is shown in Fig. 13). In contrast to the preceding case, pairs 2 and 3 are not solutions of problem (15), (16), and the critical current satisfies the inequality $\gamma_{cr}(2) > \gamma_{cr}(3)$. In other words, in the field h_2, we have a discontinuity of partial critical curve (13) of the first layer, and the jump is $\Delta\gamma = \gamma_{cr}(3) - \gamma_{cr}(2) \approx -0.001$.

Note that, at $h_B = h_2$, pair 2 turns into pair 2^a with the structure (M, Φ). At this point, the functions $\mu_{20}(h_B)$ have a vertical tangent (see Fig. 14). This means that the higher partial eigenvalues $\mu_{21}(h_B, \gamma)$ vanish; that is, there is a bifurcation of order $n = 1$ of the component $\varphi_2(x)$ of pair 2.

Similarly, for $h_B = h_3$, we also have a discontinuity of the partial critical curve caused by the difference between the maximal critical currents of the solutions on the left (3) and on the right (4). In this case, the magnitude of the jump of the current γ is an order of magnitude higher: $\Delta\gamma = \gamma_{cr}(4) - \gamma_{cr}(3) \approx 0.07$. On the interval $[h_3, h_4)$, the critical curve of the JJ is formed by a branch of pair 4. Furthermore, at the point $h_B =$

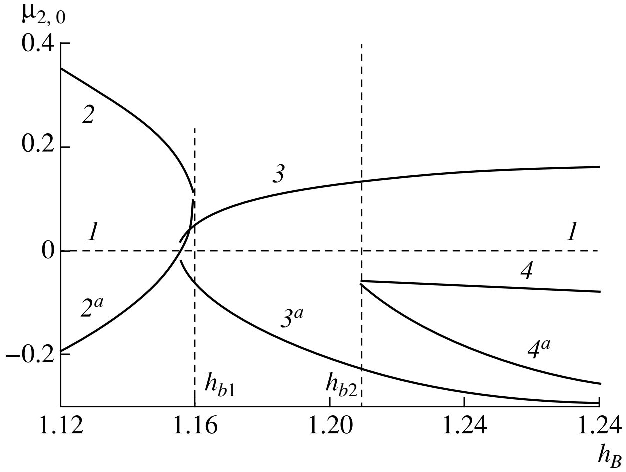

Fig. 14.

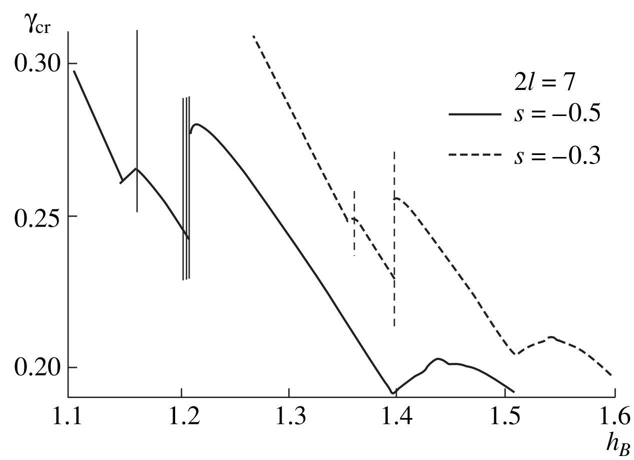

Fig. 15.

h_4, we again have a jump to the branch of the bifurcation curve of pair 3. In this case, $\Delta\gamma = \gamma_{cr}(3) - \gamma_{cr}(4) \approx -0.08$.

At the point $h_B = h_5$, there is again a jump of the critical curve from branch *3* to a branch belonging to pair 5. The magnitude of this jump is $\Delta\gamma = \gamma_{cr}(5) - \gamma_{cr}(3) \approx 0.1$.

Note that, at $h_B = h_5$, solution *5* turns into solution 5^a. The function $\mu_{20}(h_B)$ has a jump at this point (see Fig. 14); i.e., the component $\varphi_2(x)$ of pair 5 has a bifurcation of order $n = 1$.

Figure 15 demonstrates the influence of the coupling parameter between the layers s on the partial critical curve. It is seen that the critical curve moves to the right as s decreases. Thus, it tends to the position corresponding to the independent layers ($s = 0$). In the process, the structure of the jumps of the critical curve becomes simpler. For example, no solutions of form 4 were discovered in the numerical experiment, which indicates that problem (6) has bifurcations with respect to the parameter s.

6. CONCLUDING REMARKS

In this paper, the theory of partial stability is used to model the stability and bifurcations of the static configurations of the magnetic flux in the two-layered Josephson junction. It is shown that, from the mathematical point of view, the effects experimentally discovered in [5, 6] can be caused by the existence of the points of discontinuity on the partial critical curves. The location of the discontinuity points depends on the model and the values of the parameters.

Note that the methods used in this paper can be obviously applied to the case of the asymmetric model of the two-layered JJ and to the models of multilayered JJs.

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REFERENCES

1. T. L. Boyadzhiev, D. V. Pavlov, and I. V. Puzynin, "Calculation of Bifurcations of Stable States in Two-Layered Nonhomogeneous Josephson Junctions," Soobshch. OIYaI no. R5-89-173 (Dubna, 1989) [in Russian].
2. S. Sakai, P. Bodin, and N. F. Pedersen, "Fluxons in Thin-Film Superconductor-Insulator Superlattices," J. Appl. Phys. **73**, 2411–2418 (1993).

3a. L. N. Bulaevskii, M. Zamora, D. Baeriswyl, et al., "Time-Dependent Equations for Phase Differences and a Collective Mode in Josephson-Coupled Layered Superconductors," Phys. Rev. B **50**, 12831–12834 (1994).

3b. L. N. Bulaevskii, D. Dominguez, M. P. Maley, et al. "Collective Mode and the c-Axis Critical Current of a Josephson-Coupled Superconductor at High Parallel Magnetic Fields," Phys. Rev. B **53**, 14601–14610 (1996).
4. I. P. Nevirkovets, J. E. Evetts, and M. G. Blamire, "Transition from Single Junction to Double Junction Behavior in Sisis-Type Nb-Based Devices," Phys. Lett. A **187**, 119–126 (1994).
5. E. Goldobin, H. Kohlstedt, and A. V. Ustinov, "Tunable Phase Locking of Stacked Josephson Flux-Flow Oscillators," Appl. Phys. Lett. **68**, 250–252 (1996).
6. E. Goldobin and A. V. Ustinov, "Current Locking in Magnetically Coupled Long Josephson Junctions," Phys. Rev. B **59**, 11532–11538 (1999).
7. R. Kleiner, P. Muller, H. Kohlstedt, et al., "Dynamic Behavior of Josephson-Coupled Layered Structures," Phys. Rev. B **50**, 3942–3952 (1994).
8. S. N. Song, P. R. Auvil, M. Ulmer, and J. B. Ketterson, "Vortex Structure and Josephson Supercurrent in Stacked Double Josephson Junctions," Phys. Rev. B **53**, R6018-R6021 (1996).
9. T. Koyama and M. Tachiki, "*I-V* Characteristics of Josephson-Coupled Layered Superconductors with Longitudinal Plasma Excitations," Phys. Rev. B **54**, 16183–16191 (1996).
10. V. M. Krasnov and D. Winkler, "Static and Dynamic Properties of Stacked Josephson Junctions: Analytic Solution," Phys. Rev. B **56**, 9106–9115 (1997).
11. M. Tachiki and M. Machida, "Current Understanding of Josephson Plasma Theory and Experiments in HTSC," Physica C 341–348, 1493–1498 (2000).
12. V. V. Rumyantsev, "On the Stability of Motion with Respect to a Part of the Variables," Vestn. Mosk. Univ., Ser. Mat. Mekh., No. 4, 9–16 (1957).
13. V. I. Vorotnikov, *Partial Stability and Control* (Birkhäuser, Boston, 1998).
14. Yu. S. Gal'pern and A. T. Filippov, "Coupled States of Solitons in Josephson Junctions," Zh. Eksp. Teor. Fiz. **86** (4), 1527–1543 (1984).
15. K. K. Likharev, *Dynamics of Josephson Junctions and Circuits* (Gordon and Breach, New York, 1986).
16. T. L. Boyadzhiev, *Numerical Investigation of Critical Regimes in Nonlinar Field Models in Physics*, Doctoral Dissertation in Mathematics and Physics (OIYaI, Dubna, 2002).
17a. E. G. Semerdzhieva, T. L. Boyadzhiev, and Yu. M. Shukrinov, "Time-Independent Vortices in Long Josephson Junctions with Exponentially Varying Width," Fiz. Nizkikh Temp. **30**, 610–618 (2004).
17b. E. G. Semerdzhieva, T. L. Boyadzhiev, and Yu. M. Shukrinov, "Vortex Structures in Exponentially Shaped Josephson Junctions," J. Low Temp. Phys. **139** (1–2), 299–308 (2005).
18. A. N. Vystavkin, Yu. F. Drachevskii, V. P. Koshelets, and I. L. Serpuchenko, "Detection of Statistically Coupled States of Fluxons in Distributed Josephson Junctions with a Nonhomogeneity," Fiz. Nizkikh Temp. **14**, 646 (1988).
19. I. M. Gel'fand and S. V. Fomin, *Calculus of Variations* (Nauka, Moscow, 1961) [in Russian].
20. V. V. Novozhilov, *Foundations of the Nonlinear Elasticity Theory* (Leningrad, 1948) [in Russian].
21. *Bifurcation Theory and Nonlinear Eigenvalue Problems,* Ed. by J. B. Keller and S. Antman (Benjamin, New York, 1969; Mir, Moscow, 1974).
22. R. Jackiw, "Quantum Meanings of Classical Field Theory," Rev. Mod. Phys. **49** 681–706 (1997).
23. Yu. P. Rybakov and V. I. Sanyuk, *Multidimensional Solitons* (Ross. Univ. Druzhby Narodov, Moscow, 2001) [in Russian].
24. T. L. Boyadjiev and M. D. Todorov, "Minimal Length of Josephson Junctions with Stable Fluxon Bound States," Superconducting Sci. Techn. **15** (1), 1–7 (2002).
25a. T. L. Boyadzhiev, D. V. Pavlov, and I. V. Puzynin, "Newton's Algorithm for the Computation of Critical Parameters in a One-Dimensional Josephson Junction," Soobshch. OIYaI no. R11-88-409 (Dubna, 1988) [in Russian].
25b. T. L. Boyadzhiev, D. V. Pavlov, and I. V. Puzynin, "Application of the Continuous Analog of the Newton Method for Computation of Bifurcation Curves in Josephson Junctions," in *Proceedings of the Conference on Numerical Methods and Their Applications, Sofia, Bulgaria, 1988* [in Russian].
26a. E. P. Zhidkov, G. I. Makarenko, and I. V. Puzynin, "Continuous Analog of the Newton Method in Non-Linear Physical Problems," Phys. of Elementary Particles and Atomic Nuclei (JINR, Dubna) **4** (1), 127–166 (1973).
26b. I. V. Puzynin, I. V. Amirkhanov, E. V. Zemlyanaya, et al. "The Generalized Continuous Analog of Newton Method for Numerical Study of Some Nonlinear Quantum-Field Models," Phys. Elementary Particles and Atomic Nuclei (JINR, Dubna) **30** (1), 210–265 (1999).
27. V. Thomee, *Galerkin Finite Element Method for Parabolic Problems* (Springer, Berlin, 1997).
28. K. J. Bathe and E. Wilson, *Numerical Methods in Finite Element Analysis* (Prentice Hall, Englewood Cliffs, NJ, 1976).
29. V. A. Trenogin, *Functional Analysis* (Fizmatlit, Moscow, 2002) [in Russian].
30a. I. V. Barashenkov and E. V. Zemlyanaya, "Traveling Solitons in the Damped Driven Nonlinear Schrödinger Equation," SIAM J. Appl. Math. **64**, 800–818 (2004).
30b. E. V. Zemlyanaya and I. V. Barashenkov, "Numerical Analysis of Traveling Solitons in the Nonlinear Schrödinger Equation with Parametric Excitation and Dissipation," Mat. Model. **17** (1), 65–78 (2005).