Generalized Fundamental Polynomials

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TOPIC: COMPUTING

1. Interpolation with algebraic polynomials

Using the fundamental theorem of algebra it is easy to show that for every A-polynomial of degree n, the identity

$$A(x) \equiv \sum_{k=0}^{n} A(x_k) l_k(x) \tag{1}$$

holds true.

2. Interpolation with trigonometric polynomials

For every T-polynomial of order n the identity

$$T(x) \equiv \frac{1}{2n+1} \sum_{k=0}^{2n} T(x_k) \frac{\sin \frac{2n+1}{2}(x - x_k)}{\sin \frac{x - x_k}{2}} \tag{2}$$

holds true.

3. Interpolation with exponential polynomials

Correspondingly to 2. we could make a presentation for every exponential polynomial (E-polynomial) $E_n(x)$ on the basis $(1, shx, chx, sh2x, ch2x, \dots, shnx, chnx)$ or $(1, e^x, e^{-x}, e^{2x}, e^{-2x}, \dots, e^{nx}, e^{-nx})$ in the form

$$E(x) \equiv \sum_{k=0}^{2n} E(x_k) h_k(x)\,, \tag{3}$$

where the fundamental E-polynomials $h_k(x)$ can be written as follows

$$h_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{2n} \frac{sh \frac{x - x_i}{2}}{sh \frac{x_k - x_i}{2}}\,, \quad h_k(x_i) = \delta_{ki}\,.$$

4. The most general interpolation problem

Let X be a linear space of functions and $\varphi_0(x), \varphi_1(x), \dots, \varphi_n(x) \in X$. Let also $L_0, L_1, \dots, L_n$ be linear functionals defined in X. It is well known that the necessary and sufficient condition for the general interpolation problem

$$L_k(G_n) = L_k(f), \quad k = \overline{0,n}\,, \tag{4}$$

is ti have an unique solution as a generalized polynomial (G-polynomial) $G_n(x) = a_0\varphi_0(x) + a_1\varphi_1(x) + \dots + a_n\varphi_n(x)$ on the basis $\varphi_0(x), \varphi_1(x), \dots, \varphi_n(x)$ for every $f(x) \in X$ is

$$\Delta = det[L_k(\varphi_i)] \neq 0\,. \tag{5}$$

If we chose the functionals L_k as $L_k(g) = g(x_k)$, $k = \overline{0,n}$ when $x_0, x_1, \dots, x_n$ are different points in interval $[a, b]$ then the condition (5) shows that the basic functions $\{\varphi_k(x)\}_{k=0}^n$ form a Chebishev system for $[a, b]$. Many special interesting cases of a choice of $L_0, L_1, \dots, L_n$ are considered.

Lemma: *If $\bar{G}(x)$ and $\bar{\bar{G}}(x)$ are G-polynomials on the basis $\varphi_0(x), \varphi_1(x), \dots, \varphi_n(x)$ and $L_k(\bar{G}) = L_k(\bar{\bar{G}})$, $k = \overline{0,n}$ then $\bar{G}(x) \equiv \bar{\bar{G}}(x)$.*

The solution of the most general interpolation problem (4) can be written in the form

$$G_n(x) = \sum_{k=0}^{n} L_k(f) \Phi_k(x)\,, \tag{6}$$

where $\Phi_k(x)$, $k = \overline{0,n}$, are fundamental generalized G-polynomials on the basis $\varphi_0(x), \varphi_1(x), \dots, \varphi_n(x)$, for which the equalities

$$L_i(\Phi_k) = \delta_{ik}$$

hold true. This fact follows from the presentation (6) of $G_n(x)$ in the determinant form.

$$\Phi_k(x) = \frac{1}{det[L_k(\varphi_i)]} \begin{vmatrix} L_0(\varphi_0) & \dots & L_0(\varphi_n) \\ \dots & \dots & \dots \\ L_{k-1}(\varphi_0) & \dots & L_{k-1}(\varphi_n) \\ \varphi_0(x) & \dots & \varphi_n(x) \\ L_{k+1}(\varphi_0) & \dots & L_{k+1}(\varphi_n) \\ \dots & \dots & \dots \\ L_n(\varphi_0) & \dots & L_n(\varphi_n) \end{vmatrix}$$

The main result is:

For every G-polynomial the identity

$$G(x) \equiv \sum_{k=0}^{n} L_k(G) \Phi_k(x) \tag{7}$$

holds true.

The conclusion of the theorem follows from the above lemma and (6).

This result (7) generalizes the formulas (1), (2), (3) for algebraic, trigonometric and exponential polynomials.