Generalized Fundamental Polynomials

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TOPIC: COMPUTING

1. Interpolation with algebraic polynomials

Using the fundamental theorem of algebra it is easy to show that for every $A$-polynomial of degree $n$, the identity

$$A(x) \equiv \sum_{k=0}^{n} A(x_k) l_k(x)$$

holds true.

2. Interpolation with trigonometric polynomials

For every $T$-polynomial of order $n$ the identity

$$T(x) \equiv \frac{1}{2n+1} \sum_{k=0}^{2n} T(x_k) \frac{\sin \frac{2n+1}{2} (x-x_k)}{\sin \frac{n+1}{2}}$$

holds true.

3. Interpolation with exponential polynomials

Correspondingly to 2. we could make a presentation for every exponential polynomial ($E$-polynomial) $E_n(x)$ on the basis $(1, shx, chx, sh2x, ch2x, \ldots, shnx, chnx)$ or $(1, e^x, e^{-x}, e^{2x}, e^{-2x}, \ldots, e^{nx}, e^{-nx})$ in the form

$$E(x) \equiv \sum_{k=0}^{2n} E(x_k) h_k(x),$$

where the fundamental $E$-polynomials $h_k(x)$ can be written as follows

$$h_k(x) = \prod_{i=0}^{2n} \frac{sh \frac{x-x_i}{2} - sh \frac{x_i}{2}}{sh \frac{x}{2}}, \ h_k(x_i) = \delta_{i,i}.$$

4. The most general interpolation problem

Let $X$ be a linear space of functions and $\varphi_0(x), \varphi_1(x), \ldots, \varphi_n(x) \in X$. Let also $L_0, L_1, \ldots, L_n$ be linear functionals defined in $X$. It is well known that the necessary and sufficient condition for the general interpolation problem

$$L_k(G_n) = L_k(f), \ k = 0, n,$$

is that there is an unique solution as a generalized polynomial ($G$-polynomial) $G_n(x) = a_0 \varphi_0(x) + a_1 \varphi_1(x) + \ldots + a_n \varphi_n(x)$ on the basis $\varphi_0(x), \varphi_1(x), \ldots, \varphi_n(x)$ for every $f(x) \in X$ is

$$\Delta = \det[L_k(\varphi_i)] \neq 0.$$  

If we chose the functionals $L_k$ as $L_k(g) = g(x_k), k = 0, n$ when $x_0, x_1, \ldots, x_n$ are different points in interval $[a, b]$ then the condition (5) shows that the basic functions $\{\varphi_k(x)\}_{k=0}^{n}$ form a Chebyshev system for $[a, b]$. Many special interesting cases of a choice of $L_0, L_1, \ldots, L_n$ are considered.

**Lemma:** If $G(x)$ and $\bar{G}(x)$ are $G$-polynomials on the basis $\varphi_0(x), \varphi_1(x), \ldots, \varphi_n(x)$ and $L_k(G) = L_k(\bar{G}), k = 0, n$ then $G(x) \equiv \bar{G}(x)$.

The solution of the most general interpolation problem (4) can be written in the form

$$G_n(x) = \sum_{k=0}^{n} L_k(f) \Phi_k(x),$$

where $\Phi_k(x), k = 0, n$, are fundamental generalized $G$-polynomials on the basis $\varphi_0(x), \varphi_1(x), \ldots, \varphi_n(x)$, for which the equalities

$$L_i(\Phi_k) = \delta_{ik}$$

hold true. This fact follows from the presentation (6) of $G_n(x)$ in the determinant form.

$$\Phi_k(x) = \frac{1}{\det[L_k(\varphi_i)]} \left| \begin{array}{c} L_0(\varphi_0) \ldots L_0(\varphi_n) \\ \vdots \\ L_k-1(\varphi_0) \ldots L_k-1(\varphi_n) \\ \varphi_0(x) \ldots \varphi_n(x) \\ L_{k+1}(\varphi_0) \ldots L_{k+1}(\varphi_n) \\ \vdots \\ L_n(\varphi_0) \ldots L_n(\varphi_n) \end{array} \right|$$

The main result is:

**For every $G$-polynomial the identity**

$$G(x) \equiv \sum_{k=0}^{n} L_k(G) \Phi_k(x)$$

holds true.

The conclusion of the theorem follows from the above lemma and (6).

This result (7) generalizes the formulas (1), (2), (3) for algebraic, trigonometric and exponential polynomials.