Numerical study of fluxon solutions of sine-Gordon Equation

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In the frame of the Program for collaboration of JINR (Dubna) and Bulgarian scientific centers, we investigate the static regimes in the long Josephson junctions (JJs). The dynamics of the magnetic flux in a JJ of length $2l$ is described by the perturbed sine-Gordon equation:

$$\varphi_{xx} - \varphi_{tt} - \alpha\varphi_t = \sin\varphi - \gamma,\ t > 0,\ x \in (-l, l) \quad (1)$$

with boundary conditions

$$\varphi_x(\pm l, t) = h_e\,, \quad (2)$$

where φ is the magnetic flux distribution, h_e – the external magnetic field, γ – the external current and $\alpha \geq 0$ – the dissipation coefficient.

Accounting of the second harmonic in the Fourier-decomposition of the Josephson current yealds the double sine-Gordon equation. The effect of the second harmonic contribution on the properties of magnetic flux in the long JJ model has been numerically investigated in [1, 2, 3, 4].

Here, we present results of numerical investigation [5] of the the static fluxon solutions of Eq. (1) under the influence of the external magnetic field h_e in (2). Such solutions satisfy the following boundary problem

$$-\varphi_{xx} + \sin\varphi - \gamma = 0,\ x \in (-l; l),\ \varphi_x(\pm l) = h_e. \quad (3)$$

Below, we consider the case $2l = 10$ and $\gamma = 0$.

The static fluxon solutions of Eq. (1) are obtained numerically, by solving of the boundary problem (3). The stability analysis is based on numerical solution of the Sturm-Liouville problem [6]:

$$-\psi_{xx} + q(x)\psi = \lambda\psi,\ \psi_x(\pm l) = 0, \quad (4)$$

where

$$q(x) = \cos\varphi(x, p),\ p = (l, h_e, \gamma). \quad (5)$$

In this approach, the minimal eigenvalue of Eq. (4) $\lambda_0(p) > 0$ corresponds to a stable solution. In case $\lambda_0(p) < 0$ solution $\varphi(x, p)$ is unstable. The case $\lambda_0(p) = 0$ indicates a bifurcation with respect to one of parameters $p = (l, h_e, \gamma)$.

The numerical solving of Eq. (3) is based of the continuous analog of Newton's method [7]. At each Newtonian iteration the corresponding linearized problem is solved, on a uniform grids with 1025 number of nodes, using a three-point Numerov approximation of the fourth order accuracy [8].

For numerical solution of the Sturm-Liouville problem (4) we applied the standard three-point second order finite-difference formulae. First several eigenvalues of the resulting algebraic three-diagonal eigenvalue problem are obtained by means of the standard EISPACK code. Details of numerical scheme are described in [1, 2, 3] for the double sine-Gordon equation.

The known for $h_e = 0$ solutions M_0 and Φ^1 are numerically path-followed to non-zero positive h_e. The continuation process was organized as in [9].

At each ith step of the numerical continuation we analyze the stability of solution $\varphi(x, h_e^{(i)})$ and calculate the following physical characteristics:

- full magnetic flux of the distribution $\Delta\varphi^{(i)} = \varphi(l, h_e^{(i)}) - \varphi(-l, h_e^{(i)})$;
- quantity N denoted "number of fluxons" in [1] and is determined as follows

$$N[\varphi(x, h_e^{(i)})] = \frac{1}{2l\pi}\int_{-l}^{l} \varphi(x, h_e^{(i)})\, dx. \quad (6)$$

Since each solution φ of Eq. (3) is defined with an accuracy $2k\pi$ ($k \in \mathbf{Z}$) then the value $N[\varphi]$ is also defined with accuracy $2k$. The arbitrariness at the choice of integer number k can be used for the "concordance" of the value N with the value of the full magnetic flux $\Delta\varphi$ according to the condition

$$|N[\varphi] - \Delta\varphi/2\pi| \to \min\,. \quad (7)$$

Solutions φ with $n = N$, where N satisfies Eq. (7) are denoted φ^n.

Two *basic* distributions are known at $h_e = 0$: the uniform Meissner solution M_0 with $N[M_0] = 0$ and the fluxon solution Φ^1 with $N[\Phi^1] = 1$ ([1]). Minimal eigenvalue λ_0 of Eq. (4) is negative for Φ^1 and positive for M_0. As we continue basic state M_0 to $h_e > 0$ λ_0 stays positive, i.e. the branch is stable until $h_e = 2$. In the Φ^1 case, the minimal eigenvalue λ_0 crosses zero at the point $h_e = h_1 = 0.054$, i.e. the branch is unstable for $0 \leq h_e \leq h_1$ and stable for $h_1 < h_e < 2.098$. Transformation of the internal magnetic field shape of basic solutions in dependence on h_e is shown on Figs. 1, 2.

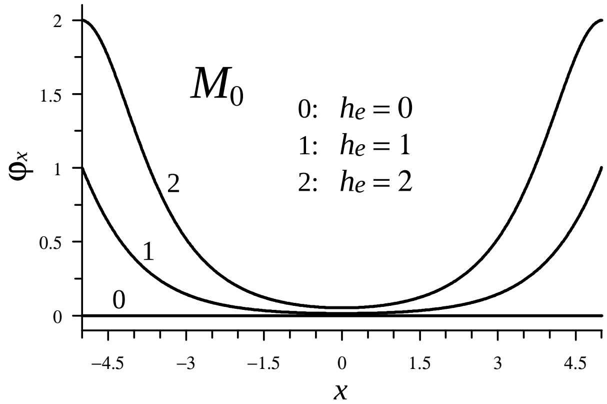

Figure 1: Internal magnetic field distribution $\varphi_x(x)$ associated with the state M_0 for several values of the magnetic field h_e.

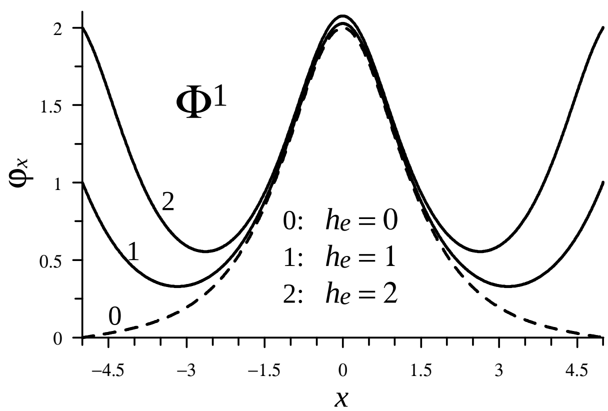

Figure 2: Internal magnetic field distribution $\varphi_x(x)$ associated with the state Φ^1 for several values of magnetic field h_e.

The $\Delta\varphi(h_e)$ branches associated with the basic solutions M_0 and Φ^1 are presented on Fig. 3. It is seen, that at some points (light circles in Fig. 3) the curves $\Delta\varphi(h_e)$ turn back to another, upper, branches. When the $\Delta\varphi(h_e)$ curve turns to the left ("$\supset$"-point) the quantity N is increased to $N+2$. So, the branch started from the basic M_0 solution at $h_e = 0$, joins the fluxons (stable and unstable) with the even N while the another branch (associated with the Φ^1 basic fluxon) connects fluxons (stable and unstable) with the odd N.

The change of stability occurs at the points (marked by dark circles in Fig. 3) where the $\lambda_0(h_e)$ curve crosses zero, see Figs. 4,5. The "$\supset$"- and "$\subset$"-turning points are indicated by the light circles. The "$\subset$"-turning points connect a pair of unstable solutions with the same number N: φ^n and $\bar{\varphi}^n$. An increase of N to $N+2$ is observed at the "$\supset$"-turning points (light circles).

Thus, for $0 < h_e < h_1$ we have a single stable static distribution (associated with the basic solution M_0). For $h_1 < h_e < h_{cr}$, $h_{cr} = 0.561$ this distribution coexists with another one associated with the basic solution Φ^1. An increasing of magnetic field h_e leads to the appearence, at h_{cr}, (most left light circle in Fig. 3) a pair of (unstable) states (φ^2, $\bar{\varphi}^2$). As h_e is growing next, the stabilization of $\bar{\varphi}^2$ occurs (most left dark circle in Fig. 3), i.e. for $h_e = 1$ we have three stable distributions to be coexisting with unstable state φ^2, see Fig. 6. Further increasing h_e induces a creation, at each "$\subset$"-point, of additional pair (φ^n, $\bar{\varphi}^n$) with growing n, see Fig. 7. At the same time, the pairs ($\varphi^n, \bar{\varphi}^{n+2}$) with previous values n are sequentially disappearing at the "$\supset$"-points.

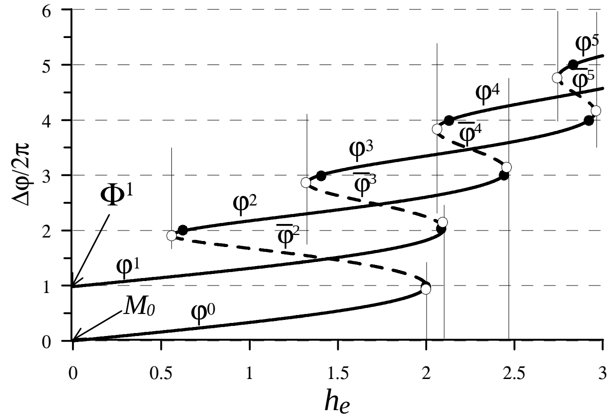

Figure 3: Dependence of the full magnetic flux $\Delta\varphi$ on the magnetic field h_e for fluxon distributions associated with M_0 and Φ^1. Solid and dashed lines correspond, respectively, to the stable and unstable states. Light circles indicate the turning points, dark circles indicate the points, where solution changes its stability.

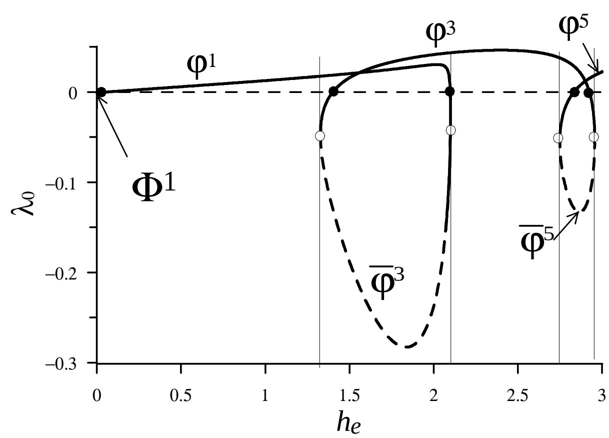

Figure 4: Dependence of the minimal eigenvalue λ_0 on the magnetic field h_e for the branch associated with Φ^1. Light circles indicate the turning points, dark circles indicate the points of stability change.

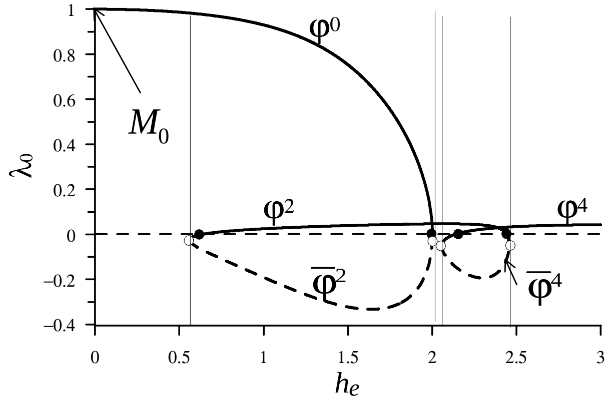

Figure 5: Dependence of the minimal eigenvalue λ_0 on the magnetic field h_e for the branch associated with M_0. Light circles indicate the turning points, dark circles indicate the points of stability change.

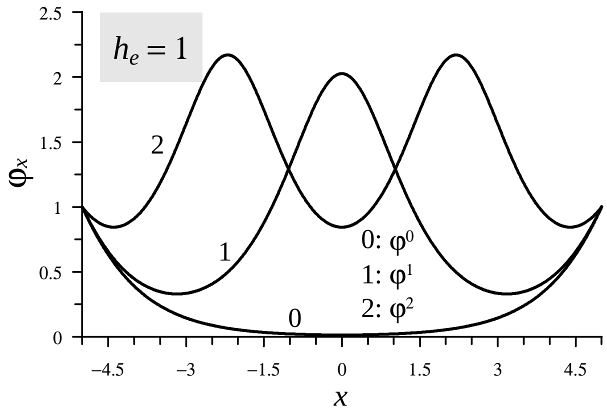

Figure 6: Coexisting stable internal magnetic field distributions φ_x at magnetic field $h_e = 1$.

The fluxon solutions of a boundary problem for the sine-Gordon equation (SGE) are investigated numerically in dependence on the boundary conditions. Interconnection between fluxon and constant solutions is analyzed. Numerical results are discussed in context of the long Josephson junction model.

The detailed information on the variation of fluxon structure with external magnetic field in long Josephson junction is very important for correct interpretation of the experimental results. Our numerical technique allowed us to establish the interconnection between the basic solution M_0 at $h_e = 0$ and the stationary distributions φ^n with even numbers n as well as the interconnection between basic state Φ^1 at $h_e = 0$ and φ^n with odd numbers n. Coexistence of different stable n-fluxon distributions at different values of external magnetic field h_e has been shown. We consider that predicted transformations of the stable fluxon distributions can be observed experimentally by investigation of the critical current in dependence of external magnetic field.

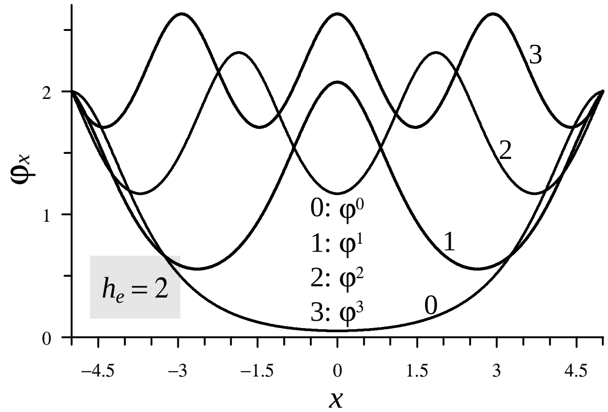

Figure 7: Coexisting stable internal magnetic field distributions φ_x at magnetic field $h_e = 2$.

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